A COMMENTARY UPON BĪRŪNĪ'S KITĀB TAḤDĪD AL-AMĀKIN

An 11th Century Treatise on Mathematical Geography

by

E.S. KENNEDY

1973

American University of Beirut

Beirut, Lebanon

CONTENTS

CHAPTER III. OBLIQUITY OF THE ECLIPTIC

CHAPTER IV. RELATIONSHIPS BETWEEN φ, ε, AND δ.

CHAPTER V. ON THE DETERMINATION OF LONGITUDINAL DIFFE

PREFACE

The book upon which this study is based was being written by Abū Rayḥān al-Bīrūnī during his journey as a political prisoner from his native Khwārazm to the capital of the Ghaznavid Empire in modern Afghanistan. This was in 1018, and he completed the work after his arrival in Ghazna.

Its primary objective is a determination of the geographical position of this city with respect to Baghdad and Mecca, thereby to calculate the direction of Muslim prayer from Ghazna. This result is achieved, and with notable precision. However, the author does not confine himself to bare computations. He feels obligated to discuss various related topics, such as the disposition of earth masses on the terrestrial globe, the causes of geological change, ancient artifacts, and so on. Techniques for the determination of latitude and longitude are described and evaluated. A basic parameter is the inclination of the ecliptic. Results obtained by other observers for it and for the coordinates of intermediate localities are cited and criticized. The end product is a self-contained treatise on medieval geodesy with numerous items of interest to historians of astronomy, mathematics and technology.

A great deal of the material is technical and cannot be understood by an offhand perusal of the text. We have sought to ease the task of the reader by putting the mathematical arguments into modern symbols and by redrawing most of the figures. All the computations have been redone and errors pointed out. Many individuals are named in the text, most of them known elsewhere in the literature. For these, references have been supplied, and biographical indications given.

The commentary is based upon the excellent critical edition of the text prepared by Dr. P.G. Bulgakov and checked by Dr. Imam I. Ahmad; topics are discussed in the commentary in the order in which they appear in the text, and references to the latter give page and line of the printed edition, separated by a colon. It is assumed that the reader will have at hand either the Arabic original or the English translation by Professor Jamil Ali. Both are listed under Taḥdīd in the bibliography at the end of this volume

The Russian translation and commentary, also prepared by P.G. Bulgakov, is listed under RT. It has been used in our study, and is referred to in various places below. Perhaps it is valid to say that RT has been written from the point of view of an orientalist, our commentary from that of an historian of the exact sciences.

Further information on the life and works of al-Bīrūnī will be found in the article under his name in the DSB. (Underlined abbreviations and short titles are references to the bibliography).

The work was done on time made available by an appointment as visiting professor to Brown University, and by grants from the National Science Foundation to the American University of Beirut. Grateful acknowledgment is made of assistance by colleagues. These include Professor Ricardo Caminos for Egyptological information, Professors Bryan Gregor, Stewart Edgell, and Mr. Sadad Husseini on geological questions, Professor Frans Bruin on medieval astronomical instruments, Mr. Dikran Keosheyan for computer programming, Professor Owen Gingerich for an application of the astrolabe, Professor David Pingree on science in India, Professor R.N. Frye on Iranian cosmology, and Professors Jamil Ali and Josef Van Ess for the interpretation of passages in the text. Mr. Zahi Khuri, Director of Publications of the American University of Beirut, has seen the book through the press with a solicitude and efficiency beyond the call of duty. Copy for offset reproduction was typed by Miss Suzy Khatchadourian.

Professor O. Neugebauer, to whom this volume is dedicated, has answered all manner of questions arising during its preparation. But, far transcending this, his example and precept have inspired and informed every scholarly task essayed by the undersigned for the past quarter of a century.

E.S.K.

LIST OF SYMBOLS AND CONVENTIONS

For easy reference, symbols used consistently in the commentary are displayed below, arranged more or less alphabetically. Where applicable and convenient they are the standard modern astronomical symbols.

The medieval trigonometric functions are, as customary, distinguished from their modern counterparts by capital initials, thus

$$\mathrm{Sin}\, x = R \sin x,$$

where R is the radius of the defining circle, usually R = 60. Where two or more such parameters are present in the same discussion, one may be shown as a subscript to avoid ambiguity, thus

$$\mathrm{Sin}_\rho\, x = \rho \sin x.$$

As is usual, sexagesimals are transcribed in ordinary numerals with sexagesimal digits separated by commas. The semicolon is used as a "sexagesimal point" except where computer output has been reproduced, where a period is utilized.

Computational results reproduced from the text can be assumed to be correct unless otherwise stated in the commentary. The reader will find a general critique of the computational methods of the Taḥdīd in Section 115 below.

All rules and explanations in the original Arabic (and in the English and Russian translations) are expressed verbally. In putting them into modern symbols we have frequently found it convenient to set up parenthetical equations within equations, say,

$$A = (\mathrm{Sin}(B = C)).$$

This usage may appear strange, but it is an accurate reflection of the text.

Where a figure in the commentary is a modernized version of one in the text, it carries the same number as its counterpart in the translation, but preceded by a C.

α	right ascension
α_{φ}	oblique ascension at a locality of latitude φ (for a definition, see Survey, p. 140).
$\frown$	arc, as $\overset{\frown}{AB}$ for the arc AB.
az.	azimuth.
β	celestial latitude.
crd	the chord function, $\mathrm{crd}\ x = 2 \sin(x/2)$.
d	half the arc of daylight; d as a superscript stands for day, not degree, which is $^{\circ}$.
$\cdot$	a dot over a symbol indicates a derivative with respect to time, $\dot{x} = dx/dt$.
Δ	difference, e.g. $\Delta x = x_2 - x_1$.
δ	declination.
e	equation in the astronomical sense, e.g. e_s is the solar equation, the difference between mean and true solar longitudes.
ε	inclination of the ecliptic, max δ_s.
φ	geographical latitude.
H	horoscope, ascendent (Survey, p. 140).
h	altitude; hours, when used as a subscript.
$\bar{h}$	zenith distance.

k	length of a degree along the meridian.
Λ	geographical longitude.
λ	celestial longitude.
m	moon, when used as a subscript.
q	equation of (half) daylight.
R	radius of the defining circle for medieval trigonometric functions.
r	rising amplitude, distance on the horizon from the east point to the rising point of the sun.

Right angles are frequently indicated on figures by a small square drawn into the vertex, e.g. angle LFC on Figure C6.

s	sun, when used as a subscript.
	ṭawq al-madār, half the circumference of the parallel of latitude through a given terrestrial point.
vers	the versed sine, $\mathrm{vers}\ x = 1 - \cos x$.
—	the vinculum, usually stands for complement, $\overline{AB} = 90^{\circ} - AB$; occasionally it indicates a mean value, or (as in Section 74) a chord.

TRANSCRIPTION OF ARABIC LETTERS

ON GEOMETRIC FIGURES

For the transliteration of Arabic words into Latin characters the standard conventions have been used. However, individual letters on the figures of the text have been transcribed as shown below.

A	ا	M	م
B	ب	N	ن
C	ص	O	ع
D	د	Q	ق
E	ه	S	س
F	ف	T	ط
G	ج	W	و
H	ح	X	ش
K	ك	Y	ي
L	ل	Z	ز

θ ث

CHAPTER I. INTRODUCTORY

1. Generalities (22:1 – 33:4)

The first chapter of the Taḥdīd, though rambling and discursive, preserves an underlying unity. It is an apologia for the study of the sciences in general, and for geography in particular, which leads eventually to an annunciation of the book's ultimate specific objective. This is to determine the geographical coordinates of Ghazna (modern Ghazni in Afghanistan), the capital city of the author's unnamed patron. From these results Bīrūnī will proceed to calculate Ghazna's qibla, the azimith of Mecca, which is the direction the Muslim faces in prayer. Along the way, divers topics are discussed, some of them of independent interest.

Among the subjects endorsed is logic, in connection with which the author blames the early translators who merely transliterated Greek technical terms (29:3–7) into Arabic characters. Had they used instead current Arabic equivalents, he says, many people who were put off by the subject would have been attracted to it. (Cf. RT, pp. 275–6.)

The name of the author of al-Masālik wʾal-Mamālik (30:5) is not given. The same title was shared by several books of a single genre. These were geographical treatises useful to the bureaucrats of an empire, systematically listing the distances between cities, the location of postal relay stations in between them, the tax assessments of the several provinces, and suchlike information. The most famous of these books was compiled by one Ibn Khurdādbih (d. 849) while he was in charge of the central postoffice at the Abbasid capital of Sāmarrā (Ḥudūd, pp. 12–15; GAL, G1, p. 258; RT, p. 276, note 60; see also Section 3 below.)

Alexander (31:6) was customarily referred to in Muslim literature as the Two-Horned. Many coins of Alexander depict him with ram's horns, the reason being as follows. After his conquest of Egypt he took his army in a disastrous march across the Libyan desert to the oasis of Siwa. There at the temple of

Zeus-Amon the priests recognized him as the ruler of Egypt. The office of Pharoah carried with it divine attributes, of which the horns of the ram-god Amon were symbolic. (Cf. EI, p. 961.)

Khālid bin al-Walīd (d. 641/2) was a famous general of the Arab armies which were brilliantly successful at the inception of Islam. The incident here referred to (33:2) doubtless occurred during one of his campaigns.

2. Adventures of a Pilot in the China Trade (33:5 - 35:4)

Persian and Arab seafarers had been trading with India and Indonesia for centuries before Bīrūnī's time, although it seems well established that they had not sailed as far as China during the Sasanian period. (Walters, p. 146). But by the ninth century voyages between the Persian Gulf and China were commonplace. Sīrāf, on the coast due south from Shīrāz, was the most important western terminus (LeStr., p. 258), and Canton (Khānfū) was the eastern one. There was a large colony of Muslim traders established at the latter place.

Collections of stories exist relating the alleged adventures of the merchants and seamen of Sīrāf and other Persian Gulf ports (e.g. Merveilles, Reinaud). Among them is one (Hourani, pp. 114-117) which, though not identical with the Māfanna tale, exhibits the same motifs: the shipwrecked sailor who refuses rescue unless he is paid and is given command of the rescuing craft, and his marvelous feats of seamanship and navigation once he is aboard. The expedient of locating one's position by the smell of particles of the bottom brought up with the lead line seems farfetched, but apparently it was part of the standard technique of fishermen off the Newfoundland Grand Banks (Captains, p. 124).

The "gates of China" (33:8) is the medieval Muslim name for the Paracel Reefs, dangerous waters in the South China Sea southeast of Hainan (Hourani, p. 72).

The tale of Māfannā is curiously echoed by a similar event which took place in the same waters during the latter part of the second World War. The United States submarine Timosa picked up ten crewmen of an American bomber which had come down in the sea south of the island of Kyushu (Morison, p. 510),

> ". . . but when the rescued aviators learned of this boat's mission they expressed a unanimous desire to return to their rubber raft and wait for a different rescue!"

The name Zābij (34:3) was sometimes used for Java (to which word it is related), sometimes central and southern Sumatra, and sometimes the whole Indonesian archipelago (EI, vol. 2, p. 575. Cf. also RT, p. 278, notes 85-93).

A mithqāl of gold weighed 4.23 grams (Hinz, p. 1).

3. First Remarks on the Qibla (35:4 - 38:19)

To determine the azimuth of Mecca from a given locality it is necessary to determine the geographical coordinates of Mecca and the second locality, and to have a sufficient command of spherical trigonometry. As for the syllogism (36:15) which Bīrūnī demolishes, it would have been correct to say that at any instant when the sun culminates at the zenith of Mecca, any observer facing the sun will be looking in the direction of the qibla for his locality. The fallacy occurs in alleging that this direction will be that of the local meridian. This latter will be true only if the observer happens to be due north or south of Mecca.

If this qualification is borne in mind, some sense can be made of the passage immediately following, 37:6-12. Let the observer determine the direction of the north celestial pole. If he then turns about toward the opposite direction he will be facing the qibla, provided that he is on the same meridian as Mecca, or thereabouts, and north of it. The word al-jadī which is clearly written in the MS (37:7 in the printed text) is the standard Arabic name for the constellation Capricorn and a star in it. It cannot mean that here; perhaps it is a scribal corruption of some alternative designation of the pole (quṭb). (Cf. RT, p. 280, note 114.)

At 38:1 is Bīrūnī's first mention of Ptolemy's geography (Geogr. in the bibliography). In this as in three other fields, astronomy, astrology and music, the power and originality of Ptolemy's work assured it primacy for centuries after its appearance.

In Section 1 above the nature of the class of books sharing the name al-Masālik has been indicated. In 38:2 the author of such a book is named. Abū ʿAbdallāh Muḥammad b. Aḥmad al-Jaihānī (fl. 920(?) was the prime minister of the Sāmānid dynast Naṣr b. Aḥmad, and as such was in a position to obtain geographical information from widely scattered sources. His work, comprising seven volumes, is lost, but it was used by many geographers after him.

Bīrūnī's efforts to obtain precise positions for geographical localities would undoubtedly be facilitated by plotting them on a

spherical surface, particularly one of the size he had made. Ten cubits diameter is about five and a half meters (Hinz, p. 55). The ease and convenience of graphical methods will be appreciated by anyone who works through the hundreds of trigonometric computations which follow in the Taḥdīd.

The calamity to which Bīrūnī refers was probably his arrest by Sulṭān Maḥmūd of Ghazna, and the time of constructing the terrestrial hemisphere was therefore during the prosperous interlude, say from 1003 to 1016 when he was in his Khwārazmian homeland and high in the favor of the Khwārazmshāh Abū al-ʿAbbās Maʾmūn (cf. RT, pp. 56-61, 280; DSB). Evidently his labors were not entirely wasted, for in the Canon he reports the coordinates of over six hundred localities.

4. The Creation, Geological Change, Fossils (38:13 - 44:8)

In connection with dates of the Creation inferred from books regarded as divinely inspired, Bīrūnī mentions, among other religions, that of the Sabians (41:1). By these he does not mean the Sabaean kingdom referred to implicitly a bit farther on at 44:9 (see Section 6). He has in mind the adherents of an important pagan sect centered at Ḥarrān on the upper Euphrates. These Ḥarrānians produced important scientists, among them Thābit b. Qurra and al-Battānī (cf. Sections 11 and 22 below). Bīrūnī was also cognizant of a second group, also called Sabians. These were the Mandeans, a Judeo-Christian sect laying great stress upon baptism. The Mandeans are still centered in southern Mesopotamia. (See EI, vol. 2, p. 21; Comm. Vol. p. 156.)

The Magians (al-majūs, 41:1) are the Mazdeans or Zoroastrians, adherents of the state religion of Iran in pre-Islamic times. In the eleventh century they probably still constituted a majority of the population in country districts. (See Comm. Vol., p. 148.)

The various Creation dates are discussed by Bīrūnī in the Canon (III, 5) and the Chron. (chap. 3). In the Taḥdīd this leads to a consideration of changes in the earth, once it has been created. The author's contemporary, Avicenna, engages in the same sort of speculation (Three Sages, p. 35), as did Aristotle (Meteorol., I, 14) much earlier.

Sīrjān (or Shīrjān, 43:7) in south-central Iran roughly midway between Yazd and Hurmuz (on the Persian Gulf), was the medieval capital of the province of Kirmān (LeStr., p. 300).

Bāb al-Abwāb (44:7, Arabic for "the gate of gates") is the modern port of Derbent (Persian darband, "a narrow pass"). Both

names indicate its strategic importance since ancient times. It commands traffic north and south along the Caspian west shore (LeStr. p. 180.)

Bīrūnī is right in stating that the regions he mentions were once sea bottoms, for their surface strata are composed of geologically recent marine deposits. Certainly he is describing fossil remains, although the "fish ears" (44:5, 15) can hardly be identified precisely. Perhaps they were some variety of sand dollar, an animal which flourishes in the sort of environment which produces cowrie shells.

5. Abū al-ʿAbbās al-Īrānshahrī (43:6, 51:4)

This interesting individual has received little notice in the literature, except in the writings of Bīrūnī. The information being largely available in Arabic only, it is useful to assemble it here.

Al-Īrānshahrī was a student of comparative religions, a believer in none save one he had invented himself, says Bīrūnī (India, transl., I, pp. 6, 249, 326), who admired his objective attitude. In this field he passes on al-Īrānshahrī's reports of Buddhist beliefs about Mount Meru and the cyclic creation of worlds, and Zoroastrian traditions concerning certain festivals (Chron., transl., pp. 208, 211.)

Al-Īrānshahrī was also interested in natural phenomena. In addition to his observations of climatic change recorded in the Taḥdīd, he recorded a curious optical phenomenon (Shadows, 15:6, 11), the double shadow cast by a man against a mountainside, and an annular eclipse observed by him in Nīshāpūr on 28 July, 873 (Oppolzer No. 4955; Canon, p. 632.)

He was a teacher of the famous al-Rāzī (Rhazes), and although none of his writings are extant, the names of two of his books, Kitāb al-Jalīl and Kitāb al-Akhīr, have survived. (See RT, p. 281, note 143; EI, vol. 3, p. 1134; Marvazī, p. 129.)

6. The Dam of Maʾrib (44:9-14)

Bīrūnī here touches upon a mass of fact deeply overlaid with an accumulation of fancy. The reference is to the ancient and powerful south Arabian kingdom of the Sabaeans (the Biblical Sheba). Its capital was at Maʾrib (or Mārib) in the southern corner of the Arabian peninsula about seventy miles equidistant from the Red Sea and the Gulf of Aden. This was indeed the site

of a great dam which broke in about 450 A.D. The Yuqṭān mentioned (elsewhere Ibn Qaḥṭān), the legendary progenitor of the Yemenites, appears in the English Bible as Yoktan (Gen. 10:28) and Yokshan (Gen. 25:3). (See EI, vol. 3, pp. 286–292; RT, p. 282, note 158.)

7. The Oxus and the Caspian (44:15 – 47:13)

This most curious passage is Bīrūnī's contribution to a misconception that persisted for two millenia--the notion that in historical times the Oxus River (the medieval Jayḥūn, modern Amū Daryā) discharged west into the Caspian instead of north into the Aral Sea as at present. However, no discredit attaches to our author for this asseveration, on the contrary, for in geologically recent times this was indeed the case, and evidence of the ancient river bed was correctly interpreted by him. But the change in direction northward occurred a million years or so ago (Nalivkin, p. 89).

The legend that the shift took place in historical times seems to have arisen as follows. The Seleucid monarch Antiochus I (reigned 293–262 B.C.) ordered a certain Patrocles to explore the Caspian. He mistook the mouth of the Atrak for that of the Oxus, and later Hellenistic geographers compounded the error by assuming the carriage of goods by water down the Oxus and across the Caspian (Tarn, pp. 112–3, 488–93; Daffiná, p. 366). Ptolemy indeed states (Geogr., VI, 9), as Bīrūnī says (45:3), that the Oxus discharges into the Caspian. After his time other Muslim savants helped to perpetuate the story, whence it was taken up by European orientalists (LeStr., p. 456.)

The situation may yet be reversed. Peter the Great proposed a waterway east from the Caspian. Much later, in 1873, the project was seriously investigated and pronounced feasible, but no actual construction was undertaken. In 1951 the Soviet government commenced work on a canal to connect the Oxus with the Caspian port of Krasnovodsk via the Özboi valley. The project embraces flood control, irrigation, and power development. The feeder canal from Takhiya-Tash in the upper delta of the Oxus was opened in 1953, but no progress has been announced since, and the project may have been abandoned (Elne, vol. 1, p. 456; Wheeler, pp. 172–3.)

8. Places and Peoples (44:15 – 47:13)

Jurjān (45:1, modern Gurgān, cognate of the ancient Hyrcania) is the name of a city and the region surrounding it at the southeast corner of the Caspian. The town is on the Atrak River about fifty miles east of the sea. (LeStr., p. 377.)

Khwārazm (45:1) is the ancient name of the region south of the Aral and watered by the Oxus. It has been the seat of advanced civilization since high antiquity. It was Bīrūnī's homeland. Its capital was Kāth, which Bīrūnī refers to only as "the city of Khwārazm" (79:22). The site now bears Bīrūnī's name. It is just north of Khīva, on the opposite side of the Oxus (LeStr., p. 446).

Balkh (45:2, the modern Mazār-i Sharīf, LeStr., pp. 420–423) lies just south of the middle reaches of the Oxus. In the ninth century it was the greatest city of Khurāsān.

Balkhān (45:3, LeStr., pp. 455–7) is supposed to have been located in the vicinity of the headwaters of the Atrak.

Zamm (modern Kerki) and Āmūya (modern Chardzou) are both on the left bank of the Oxus, downstream from Balkh (RT, p. 283, notes 167,8; LeStr., p. 403).

That the Khazars (45:8) should be mentioned here is strange. Bīrūnī is describing a mouth of the Oxus on the eastern shore of the Caspian, whereas in his time the Khazars were established to the west of it, their lands reaching the Don and the Volga on the north and the Crimea to the west (Hudud, pp. 435, 450–460).

The Ghuzz (46:1, Arabic for Oghuz) were the great Turkic people from whom came the Saljuq and Ottoman dynasties. By the tenth century they had infiltrated the steppe off the southeastern corner of the Caspian; but their territory extended much farther, right around the north of the sea to the Volga. Turcoman (47:10), or Türkmān, is synonymous with Ghuzz (Hudud, pp. 311–2, 435; EI, vol. 2, p. 168).

The Lion's Mouth (46:2) we cannot spot with certitude. The name figures in the Iranian epic Shāhnāmeh, in Kay Khusro's sea chase of the Turkish emperor Afrāsiyāb. But whereas the royal pursuer embarks upon what should be the Caspian, the Lion's Mouth, in the curious geography of the epic, turns out to be a whirlpool in the sea of China (Hādī Ḥasan, pp. 9–12). It is probably a narrow gorge in the lower reaches of the Oxus (RT, p. 284, note 172; Togan, p. 56, note 1).

Fārāb (46:5, modern Utrar) is on the Jaxartes (modern Sirdaryā) about 400 miles east of the Aral.

Fuḥma (46:6, or Fakhmī, etc.) is a locality somewhere in the delta of the Oxus (RT, p. 284, note 176).

Mazdubast (46:10, 47:11) is a dry valley through which in

ancient times the water of the Oxus was supposed to have been conveyed to the Sary Kamish depression southwest of the Aral. The resulting lake was what Bīrūnī calls the Sea of the Virgin. (47:13). (See Togan, p. 57, note 1; RT, p. 284, note 181).

The Bujnakians (46:9, 47:2) were a Turkic people known in Europe as the Pecheneg, although the name has many forms: Pačnak, etc. They were of Central Asiatic origin, and it is of interest that Bīrūnī should pass on information of their once having lived east of the Caspian, for by his time they had long since been pushed west. In the ninth century they were in the region between the Don and the Dnepr. (See EI, vol. 3, p. 1036; Ḥūdud, pp. 312-5, 435.)

Alān and Ās (47:2, delete the al preceding the second name as the Arabic definite article) are synonyms for a single people which, like the Pecheneg, moved west of the Caspian long before Bīrūnī's time. Driven in front of the Huns in the fourth century, groups of the Alans entered Gaul, thence, with the Vandals they passed over into North Africa. Other elements, the Ossetians (= Ās) remain in the Caucasus to this day. Their language, an example of the Iranian group of the Indo-european family, has been studied intensively by Soviet linguists. (See EI, vol. 1, p. 311-2.)

9. Earthquakes and Floods (48:1 - 49:4)

Abū al-Faḍl ibn al-ʿAmīd (48:1, fl. 950), prime minister of the Buwayhid ruler Rukn al-Daula, was also a man learned in several branches of scholarship. His works have not survived (GAL, S1, p. 153). Rūyān was the name of a mountainous district near the southwest corner of the Caspian (LeStr., p. 373).

Antioch has been visited by many earthquakes, the most disastrous being that of 526 A.D., when a quarter million inhabitants are said to have been killed. This was not the one referred to by Bīrūnī, which occurred in November of 268, which was indeed the second year of Justinian. The precise location of Claudiopolis (Qalūdhia, 48:8) has not been determined, but it was near Malatya on the Anatolian plateau. A blocking of the Euphrates by a landslide is dated by Syrian chroniclers at 1152, the century following Bīrūnī, but the incident so greatly resembles the one relayed by the Taḥdīd as to cause suspicion that some common Syriac source garbled the date.

In Meteorol. I, 14 Aristotle writes substantially what is attributed to him in 48:12 - 49:4.

10. The Ancient Nile - Red Sea Canal

In recent geological periods (pliocene or pleistocene), the Gulf of Suez reached considerably farther north than at present, and a branch of the Nile apparently discharged into it through the Wādī Ṭūmīlāt, having left the main channel in the vicinity of what is now Cairo. In the course of time the eastern delta rose gradually, stopping perennial flow through the wādī, causing the head of the gulf to move south, and leaving isolated bodies of water, the Bitter Lakes, in the intervening depressions. The latter part of this movement took place in historical times (and may still be continuing). To maintain irrigation in the wādī, by then cultivated, it became necessary to deepen the ancient channel, turning it gradually into a canal. Some of these operations may have been undertaken during the reigns of Sesostris I or Sesostris III (Twelfth Dynasty, 1971-1978 and 1878-1843 B.C. respectively). In the course of time these two monarchs were conflated into a single legendary Sesostris to whom many public works were indiscriminantly attributed.

Thus far the only artificial watercourse involved was one following approximately the present Sweet Water Canal as far as the region of Lake Timsāḥ. The project of extending it to the Red Sea may have been undertaken and completed by Neko II (26th Dynasty, 610-595 B.C.). It was certainly either reconstituted or completed under the Iranian Achemenian dynasty, for four stela of Darius I (521-486 B.C.) along the trace of the canal record its excavation and the passage of vessels bearing tribute through it en route to Fārs.

At whatever time it occurred, opening of a channel between the Bitter Lakes and the Red Sea would cause a rush of the waters of the latter into the former. This is because evaporation insures a drop in the level of the lakes unless they communicate with the sea. This phenomenon may be the origin of the fear, repeatedly expressed in the ancient sources, that the canal would cause the contamination of the Nile delta by salt sea water. (See Posener.)

Ptolemy II (reigned 285-247 B.C.), not Ptolemy III (247-221), carried out further works on the canal, evidently to insure that the fresh water from the Nile would reach the port city at the Red Sea end, and involving some sort of lock there.

In Roman imperial times the waterway was reopened by Trajan and Hadrian (c. 130 A.D.). By this time the port at its mouth had taken on the name Clysma, whence the Arabic designation Qulzum for the Red Sea. The canal continued in use into Byzantine times, but by 617, when the Sasanian Khusro campaigned in the region, it had again been abandoned. After the Arab conquest of Egypt it was

reopened at the order of the caliph ʿUmar (642) and used to ship Egyptian wheat to the Ḥijāz. Its final closure is reported to have taken place in about 765 when the caliph al-Manṣūr is supposed to have stopped its mouth as a means of bringing pressure upon rebels holding out in Medina. (See Bourdon.)

Many ancient and medieval authors mentioned the canal: Herodotus, Diodorus Siculus, Strabo, Pliny, al-Farghānī, al-Kindī, al-Masʿūdī, and others. It is of interest to ask where Bīrūnī received his information on the subject. In the passage of the Taḥdīd just preceding his report on the canal he has referred to Aristotle, so it is natural to consult the latter. Indeed the Meteorol. I, 14 states pretty much what Bīrūnī does down to the remark about Ptolemy, the Ptolemaic dynasty being posterior to Aristotle. Diodorus and Pliny both name Ptolemy II, the former stating that the matter of the difference in levels was solved with a lock. Strabo says more vaguely "the Ptolemaic kings" (Bourdon, pp. 2-4).

No other extant source connects Archimedes with the affair. His life spanned the reigns of both Ptolemy II and Ptolemy III, and he is supposed to have been in Egypt (Heath, vol. 2, p. 16), so there is nothing inherently impossible about the report.

None of the other sources say anything about the Romans having destroyed the canal in fear of the Persians. The legend may somehow be connected with the expedition of Khusro mentioned above.

The passage in the Taḥdīd about the canal was paraphrased (and badly garbled) in the cosmography of Ibn Ayās written in 1516. It was in turn published and translated into French in 1810, this being by all odds the earliest European notice of the Taḥdīd (Langlés, pp. 13, 17).

(See RT, pp. 286-7, notes 206-8.)

11. Deserts and Ancient Remains (50:1 - 55:2)

The Karkas (or Kargas) mountains (Pers. kūh = mountain) are east of Kāshān along the edge of the great salt desert in the middle of the Iranian plateau (LeStr., p. 208). It is to this desert which the passage refers. The lake Zarah (50:6) is in the region of Seistān (Sijistān) straddling the southeast boundary of modern Iran (LeStr., p. 338). Afrāsiāb (50:5) was the ruler of the mythical Turkish opponents of the Iranians (cf. Section 8 above).

Baṣra (51:2) is on the right bank of the Tigris near where it enters the head of the Persian Gulf.

For al-Īrānshahrī, see Section 5 above.

Nīshāpūr (51:5, or Naysābūr), capital of the western quarter of Khurāsān and home city of ʿUmar Khayyām, is near modern Meshed (Mashhad) in eastern Iran (LeStr., pp. 382-7).

Concerning Jurjān (51:10), see Section 8 above.

The passage referred to (52:7) in Aristotle is Meteorol. I, 14 (p. 113) or II, 1 (p. 125).

Thābit b. Qurra (826-901, Suter, pp. 34-38) was a great scientist and a Sabian (see Section 4 above). Many of his works are extant, but we cannot locate the passage here referred to (53:12). Concerning the salinity of the sea, cf. Meteorol. II, 1-3.

12. The Situation of the Southern Hemisphere (55:3 - 58:23)

Here the reference (55:4,12) is doubtless to Meteorol. 2.5, pp. 179-185, although we have been unable to find the precise formulation which Bīrūnī seems to quote. Aristotle apparently feels that considerations of symmetry ensure that the disposition of land masses in the southern half of the globe shall resemble that in the north. Bīrūnī also appeals to this principle in arguing that an elevated circular shoulder in the north should have its counterpart in the south, the resultant shape approximating a cylinder rather than a sphere (56:5).

The passage concerning the differential drying of inkdrops on a paper illustrates Bīrūnī's readiness to experiment, however faulty his explanation may have been.

For the material beginning with 56:20 it is necessary to supply a certain amount of technical background. In order to account for the variable angular velocity with which the sun's projection travels along the ecliptic it was customary to assume that the actual sun rotates with constant speed in a circular orbit with the earth inside the orbit, but displaced from its center. The amount of the eccentricity was about a thirtieth of the mean earth-sun distance, so that at apogee (greatest distance) the sun would be about a fifteenth part farther from the earth than at perigee (least distance). Furthermore, it was thought that this annual approach of the sun to the earth would cause an annual and marked increase in the heat received from the sun (or in the evaporation it causes, 57:2). Bīrūnī calculated (Canon. p. 693) that the solar apogee was in his time near $\lambda = 85^{o}$, only five degrees removed from the summer solstice ($\lambda = 90^{o}$), hence the perigee was five degrees from the winter solstice ($\lambda = 270^{o}$). But at the summer solstice the sun is nearest the zenith at culmination, and farthest from it at the winter solstice. Hence the paragraph beginning with 56:19.

Most Islamic astronomers believed that the solar apogee was fixed with respect to the fixed stars, which implies that its longitude slowly increases with the motion of precession. It is to this that Bīrūnī alludes in 59:4. Ptolemy, on the other hand, regarded the apogee as fixed with respect to the vernal point, hence having constant longitude (<u>Exact Sc</u>., p. 192). Bīrūnī reports this divergence of opinion in 58:13. He also remarks (57:17) that the eccentric (or epicyclic) model was developed in order to enable the astronomer to predict the sun's variable angular velocity. It may be regarded as an abstract device only, which does not require the actual sun to approach and recede from the earth.

The Abū Jaʿfar al-Khāzin (d.c. 965) mentioned in this connection (57:21) was a rather obscure scientist of Khurāsān (<u>Suter</u>, p. 58). He is not to be confused with ʿAbd al-Raḥmān al-Khāzinī, who lived later. Abū Jaʿfar's ideas are all implicit in Ptolemaic planetary theory.

If the notion of apsidial increase in longitude is accepted, then it indeed follows that in the course of time the perigee would eventually precess to the vicinity of the summer solsticial point, just the opposite of its situation in Bīrūnī's time, causing perigee to culminate near the zenith in the northern hemisphere. It was predicted that when that happens evaporation would dry up the seas occupying (as was thought) the southern hemisphere and flood the northern (57:7).

Bīrūnī perhaps takes more pains to demolish this theory than it is worth. He says that (1) because of the effect of the daily rotation the theory demands that in his time all the earth be dry land along a zone of the northern hemisphere, which is contrary to the facts (57:16). Further, (2) the actual earth-sun distance may not vary at all (57:17), as remarked above. Next (3), the slow oscillation in the water between the two hemispheres would alter the earth's centroid (58:7, for Ibn al-ʿAmīd, see Section 9 above). Finally (4), the apogees may not move at all (58:13).

13. Zones Permitting Habitation (59:1 - 62:13)

To examine possibilities in the southern hemisphere Bīrūnī considers known extremal but habitable situations in the northern hemisphere and seeks to duplicate them south of the equator. It is tacitly assumed that only two considerations affect the temperature of a locality: (1) the distance from the sun to the earth (see Section 12 above) and (2) the sun's nearness to the zenith at meridian passage.

To maximize (1) the sun should be at its closest to the earth, i.e. at perigee, which happens to be near the winter solstice. As for (2), the best that can be done is to choose a northern hemisphere region which is as far south as possible and still known to be habitable--the first climate.

Now the latitude of the middle of the first climate (see Section 44 below) from the table on p. 141 of the edition is $\varphi = 16;39^\circ$, rounded off to minutes. Under these circumstances the meridian zenith distance of the sun will be (see Figure A)

$$\bar{h} = \varphi + \varepsilon = 16;39^\circ + 23;35^\circ = 40;14^\circ \approx 40^\circ ,$$

as the text says in 59:6. The value of ε used above is that settled upon by Bīrūnī in 116:2.

A locality in the southern hemisphere which has the same zenith distance at the same earth-sun distance (i.e. winter solstice in the <u>northern</u> hemisphere, $\lambda = 270^\circ$) should have the same climatic conditions. Its (southern) latitude (see Figure B) will be

$$\varphi_1 = \bar{h} + \varepsilon = 40;14^\circ + 23;35^\circ = 63;49 \approx 64^\circ ,$$

as the text says in 59:8. For this locality at the time of the (northern) summer solstice ($\lambda_s = 90^\circ$) the solar zenith distance at culmination will be

$$\bar{h}_1 = \bar{h} + 2\varepsilon = 40;14^\circ + 47;10^\circ = 87;24^\circ ,$$

which does not round off to the text's 84° (58:12, cf. <u>RT</u>, p. 290, notes 263-5). Nevertheless Bīrūnī's next statement is valid, that no place on the northern hemisphere can be found having this meridian zenith distance at the summer solstice. The greatest such zenith distance possible is $\bar{\varepsilon} = 66;25^\circ$ (cf. 59:15).

For a place of northern latitude 60° (59:17) the maximum solar zenith distance at culmination will be (Figure A)

$$\bar{h} = \varphi + \varepsilon \approx 60^\circ + 24^\circ = 84^\circ ,$$

but this will occur only at the winter solstice ($\lambda_s = 270^\circ$), when the sun is nearest the earth. Even so it is known that this latitude is too cold to be habitable (59:19). Hence living things cannot subsist at a southern latitude of 64° or greater (60:2).

The remarks about a southern locality of latitude 48° ($\approx 2\varepsilon$, 60:6) do not seem to make sense. Perhaps what was intended is that at such a place the climate becomes warmer than the coolest season on the equator and colder than the coldest temperature reached at a northern latitude of 48°.

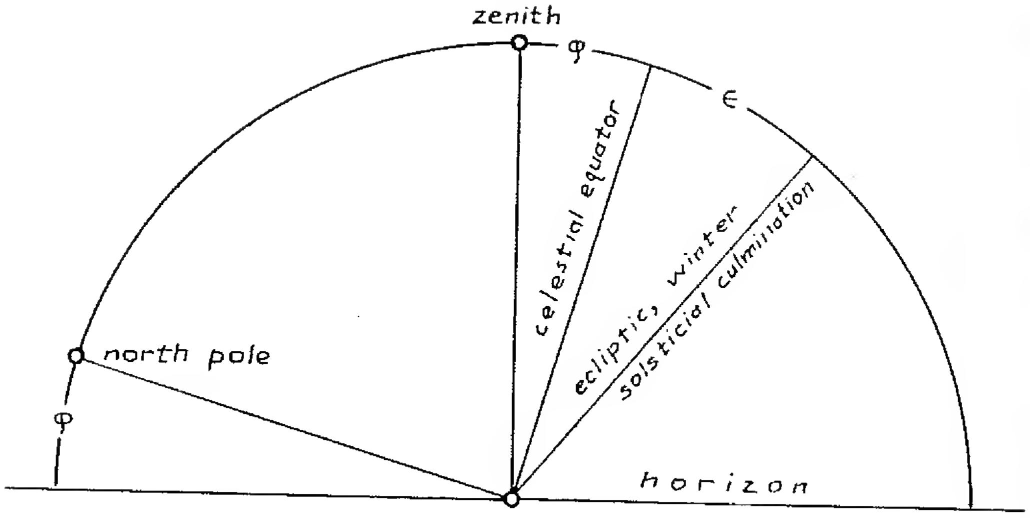


Figure A

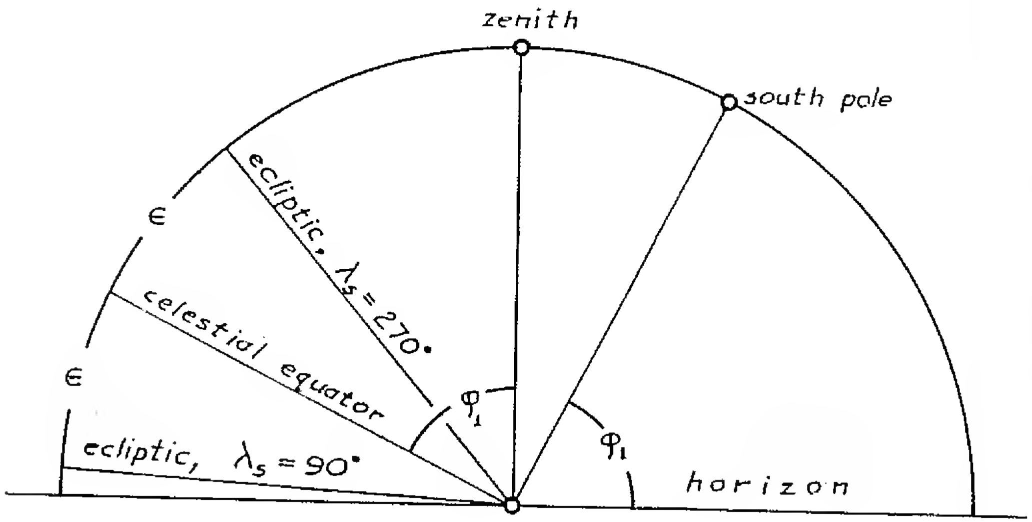


Figure B

In any event, the final conclusions (61:11) are clear, that since in the northern hemisphere the two effects work against each other, the sun approaching the earth in the winter and receding in the summer, hence the result is a tempering of the extremes of heat and cold in this hemisphere.

Bīrūnī now reverts to a possibility touched on before—that the shifting of large earth or sea masses may cause a shift of the earth's centroid, hence a change in the tilt of the earth's axis. For this reason terrestrial latitudes should be continually redetermined (61:17). This leads naturally to the chapter's peroration: the statement that his ultimate objective is the determination of the qibla at Ghazna, and the invocation of divine assistance.

14. Determination of Local Latitude by Observing a Never-Setting Star (63:1 – 65:11, 67:5–18)

By reference to our Figure C1, an adaptation of the first part of Bīrūnī's Figure 1, the reader will readily follow the burden of the argument in this passage. We note that the polar distance of a never-setting star, whose day-circle is TD, is the complement of the star's declination, $\bar{\delta}$|. To determine the local latitude (φ), observe the star's maximum altitude (TG) and its minimum altitude (DG), and take their arithmetic mean.

If we let TD expand until it passes through the zenith A while maintaining D above G, the situation will be that represented in the second part of Figure 1, and T will coincide with A. The same rule as before will again yield φ as remarked in 65:8, but Bīrūnī says (65:2) $\varphi = (DA/2) + DG$. This is correct, for then $DA = 2\bar{\delta}$, and the rule is $(2\bar{\delta}/2) + DG$, which is indeed φ. Alternatively, he gives the rule (65:3) $\varphi = (DG/2) + 45^\circ$. Now $\bar{\delta} = \bar{\varphi}$, so $(DG/2) + 45^\circ = ((\varphi - \bar{\delta})/2) + 45^\circ = ((2\varphi - 90)/2) + 45^\circ = ((2\varphi - 90^\circ)/2) + 45^\circ = \varphi$, again all right. Bīrūnī's proof is unnecessarily involved, and it seems hardly worthwhile attempting to reconstruct it.

If the parallel circle TD expands still more so as to pass beyond the zenith (without crossing the horizon), the situation will be that depicted in the third part of Figure 1. Now $\bar{\delta}$ is half the sum of DA and AT (64:10), whereas in the first two cases it is half their difference.

Referring again to Figure C1, let D, the low point on the star's day-circle be depressed until it coincides with G, the north point on the horizon (67:15–18). The situation will then be that displayed in Figure 2 of the text, the three situations shown there occurring when the point of culmination T is (1) north of the zenith A, (2) coincident with A, and (3) south of it. The calculation, such as it is, is the special case of that discussed above, with DG = 0. In all three situations, $\varphi = TD/2$. In case (2), $\varphi = 45^\circ$.

15. Application by the Banū Mūsā: the Latitude of Baghdad (66:1 – 67:5)

A certain Mūsa b. Shākir, a reformed highwayman from Khurāsān, produced three sons who became well-known scientists (Suter, p. 20). Two of them carried out the observations discussed in this passage.

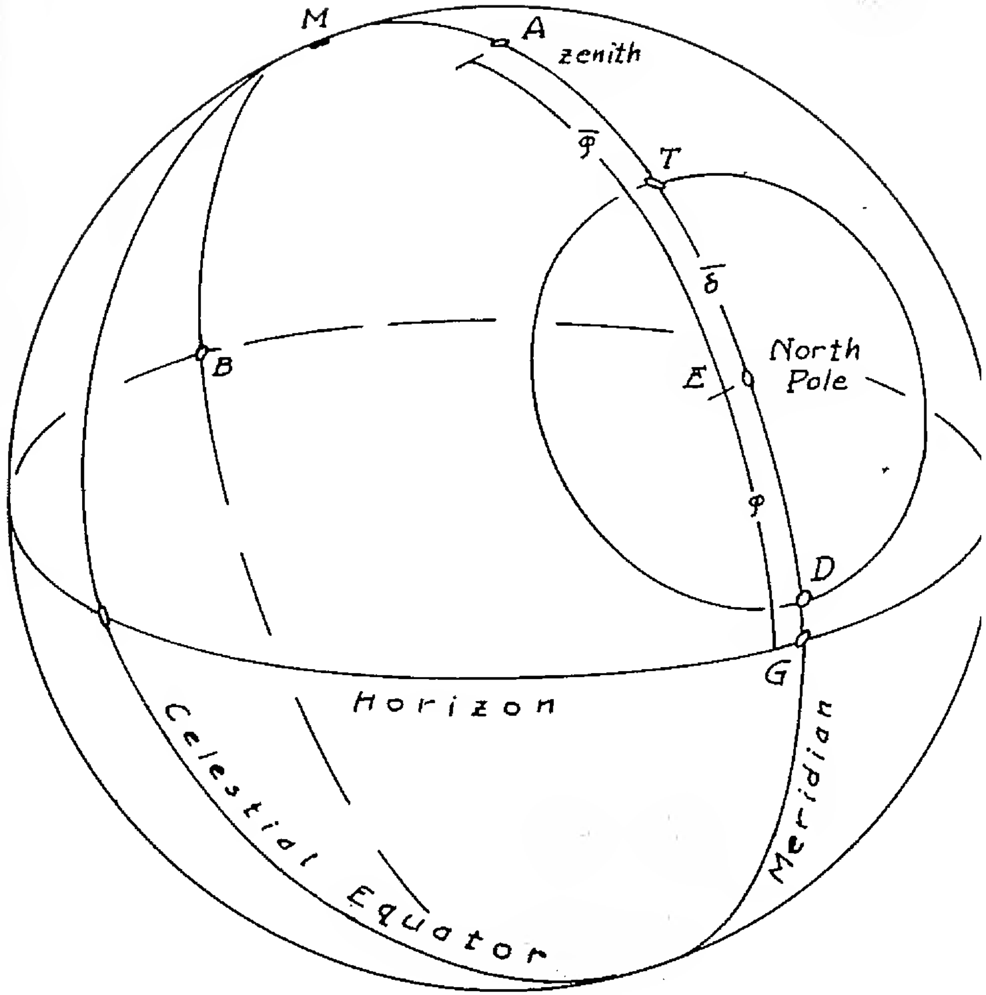


Figure C1

(1) Observation of the first star of the three, UMa δ(66:3) gave

$$\begin{aligned} TG &= 60;46^\circ \\ DG &= 6;5^\circ \\ TG - DG &= 54;41^\circ = TD = 2\,\bar{\delta} \\ TD/2 &= 27;20,30^\circ = \bar{\delta} = ED \\ \delta &= 62;39,30^\circ \quad \text{(not calculated in the text.)} \\ ED + DG &= 33;25,30^\circ = \varphi, \quad \text{for the latitude of} \end{aligned}$$

Baghdad (66:8).

(2) The second star is UMa γ(66:10), giving

$$\begin{aligned} TG &= 63;13^\circ \\ DG &= 3;45^\circ \\ 2\cdot\bar{\delta} &= 59;28^\circ \\ \bar{\delta} &= 29;44^\circ \\ \delta &= 60,16^\circ \quad \text{(not calculated in the text.)} \\ \varphi = DG + \bar{\delta} &= 33;29^\circ. \end{aligned}$$

(3) The third star is UMa ζ (66:13)

$$\begin{aligned} TG &= 62;3^\circ \\ DG &= 4;48^\circ \quad \text{(restore the printed text to this value.)} \\ 2\,\bar{\delta} &= 28;37,30^\circ \\ \delta &= 61;22,30^\circ \quad \text{(not calculated in the text.)} \\ \varphi = DG + \bar{\delta} &= 33;25,30^\circ \end{aligned}$$

The modern value for the latitude of Baghdad is 33;20°, so that all these determinations are high. The variant reading for the third star is

$$\begin{aligned} TG &= 62;13^\circ \\ DG &= 4;48^\circ \\ TG + DG &= 67;1^\circ = 2\varphi \\ \varphi &= 33;30,30^\circ, \quad \text{which is worse than the other two.} \end{aligned}$$

The year 248 A.H. began on 7 March, 862, while 232 Yazd. began on 20 April, 863. So the two years do not overlap at all. At any rate the vicinity of 862 A.D. is intended.

16. Three Techniques for Determining Local Latitude When the Observed Day-Circle Intersects the Horizon (68:1 – 72:13)

The first method is illustrated in Figure C3. It involves three rods of equal length, pivoted together at E on a horizontal plane upon which the north-south line EB has been marked as shown.

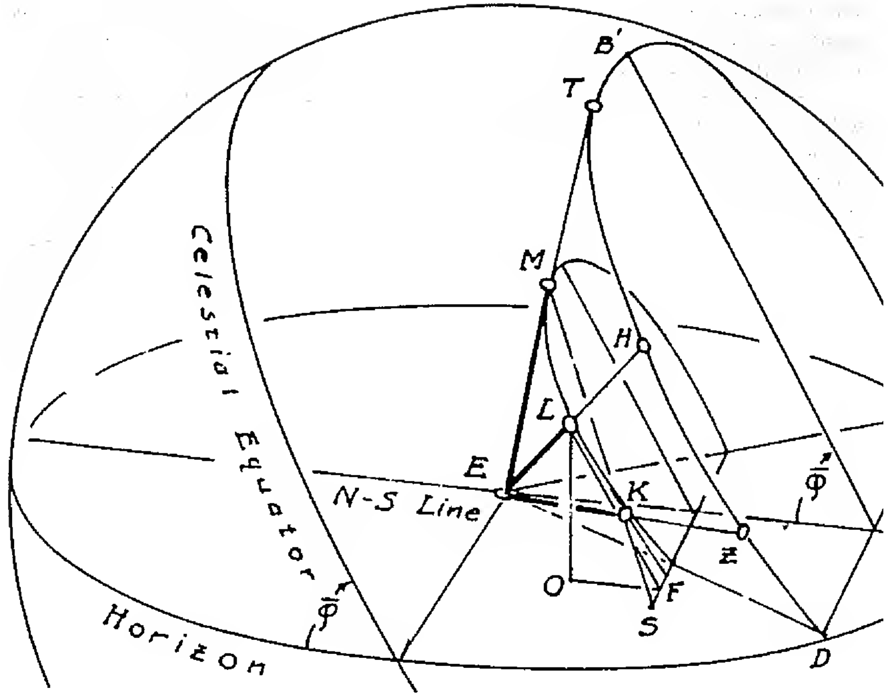


Figure C3

The lettering is that of the text's Figure 3, except that we are unable to find any single point B consistent with the several references to it, and B' is the culminating point of the star being observed. Each rod is sighted on the star at a time when it is above the horizon. The line, say KM, determined by the endpoints of any two of the rods is extended to meet the horizon plane at S. By means of a plumbline find the foot (O) of the perpendicular dropped from any one of the endpoints, say L. Draw SF perpendicular to EB; OF perpendicular to SF, and join L to F. Then in the right triangle OFL angle F is $\bar{\varphi}$, the complement of the local latitude.

Figure 3 in the manuscript is faulty. The text mentions threads extended between L and K, and also M and K, and it is not clear whether S is to be where MK meets the horizon, or LK. The matter is immaterial; either will do, and in fact the third rod is redundant. It would not be redundant if the north-south direction were assumed unknown at the beginning of the observations, for then

the two points in which MK and LK extended meet the horizontal plane would determine the east-west direction, which is needed to complete the determination.

If the observer is on the equator, $\bar{\varphi} = 90^\circ$, and the planes of all the day-circles will intersect the horizon orthogonally. Under these circumstances, as Bīrūnī remarks (70:1), the feet of the verticals from the three endpoints will be collinear. He calls these verticals "sines", and if the common length of the rods is thought of as the radius of the defining circle for the sine function, each vertical, e.g. OL, is indeed the sine of the altitude of the rod from which it depends (here EL).

The second and third techniques (71:4 - 72:13) are closely related. Both involve observations of the sun, not a fixed star, and both employ a large, fixed spherical surface. For the second method a hemisphere with horizontal base is prescribed; for the third a whole sphere, but in the latter case only the upper hemisphere is actually used. This having been arranged, a device is required which will determine, for any daylight instant, the piercing point on the sphere of that ray of the sun which then passes through the center of the sphere. The methods differ only to the extent the two objects differ which are sketched in section on Figure C3A.

The one on the right is reminiscent of an ice cream cone with an orifice at its vertex. When placed with the edge of its lateral surface on the sphere, the axis of the cone will pass through the center of the sphere. The cone is to have a circular plate, pierced at its center, and fixed as shown so that it is normal to the axis and just touches the spherical surface. The cone must also have one or more holes in its lateral surface, near the base and sufficiently large as to permit the observer both to see the upper part of the base plate and to insert a marking device through its hole. It is operated as follows: Move the cone so that the ray of sunlight which enters the vertex falls upon the hole in the center of the base plate. Mark the point on the sphere immediately below this hole. The mark will be the map on the sphere of the sun's position at the instant the observation was taken. Repeat this operation twice more in the course of the day. The resulting three marks determine on the sphere the sun's daily path for that day. Determine its pole. The altitude of the pole is the local latitude.

The device on the right is a straight rod having a flat circular base fixed at one end, normal to the rod. The bottom of the base is concave. Hence when it is placed on the spherical surface, the rod, like the axis of the cone above, will be the prolongation of a radius of the sphere. Slide the base about until the rod has no shadow in the sunlight. The rod will then be pointing at the sun.

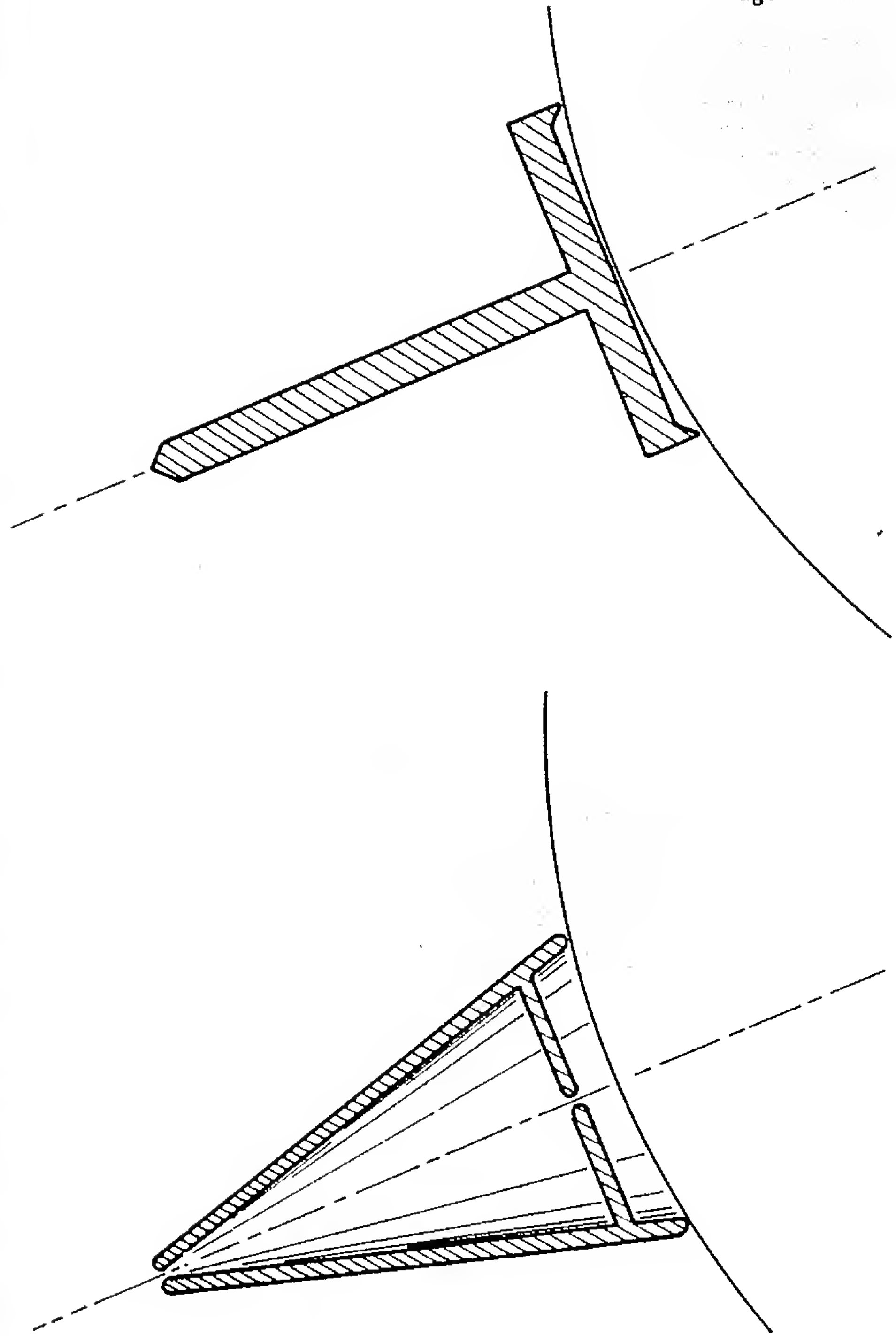


Figure C3A

Mark around the edge of the base. The center of the resulting small circle will be the map of the sun for that time. Find two more such points, and proceed as described above.

All three of these arrangements are quite impractical if considered as methods of obtaining accurate latitude determinations. Especially for the first, the technical problem of mounting three sticks upon concurrent universal joints would be well-nigh unsolvable. The second and third methods might be useful for demonstrating some basic facts of spherical astronomy.

17. Local Latitude Determination from Two Solar Observations, Worked Examples for Jurjāniya (72:13 – 82:9)

This involves the calculation of φ from horizon coordinates of the sun observed twice during the day. The situation is illustrated on Figure C4. The method in modern symbols is displayed below, each expression followed by the numbers in the worked example for Jurjāniya observed on Friday, 7 December, 1016 (75:9), for which $h_1 = 21;10^\circ$, $az_1 = 67;30^\circ$, $h_2 = 14;50^\circ$, and $az_2 = 52;30^\circ$.

By similar triangles,

(73:9) $$\frac{EO\ (=\mathrm{Cos}\ h_1)}{OC} = \frac{EM\ (=R)}{MS\ (=\mathrm{Sin}\ az_1)}$$

from which OC can be calculated. In particular

$$OC = \frac{(\mathrm{Sin}\ az_1)(\mathrm{Cos}\ h_1)}{R} = \frac{55;25,58 \times 55;57,7}{60}$$

(75:15) $$= \frac{40,198,369,266 \text{ fourths}}{60} = 51;41,35,$$

in which all computations are precise save that the last digit of OC should be 34.

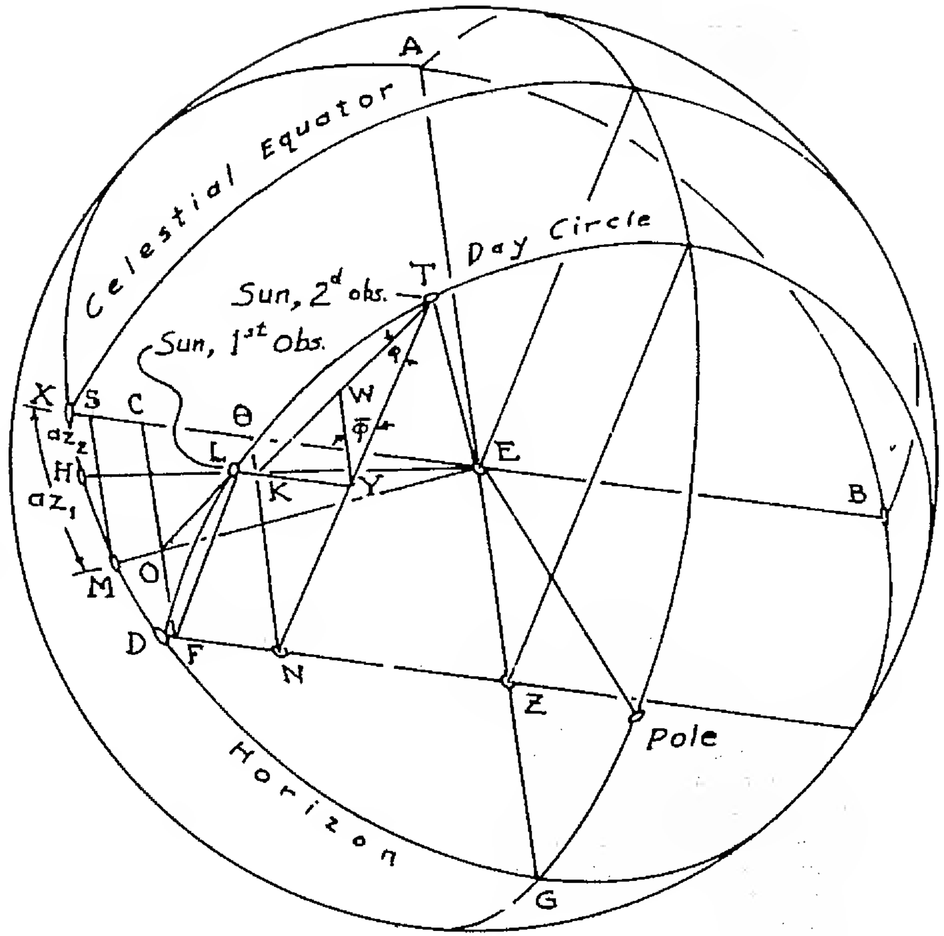


Figure C4

In like manner,

$$\frac{EK\ (=\text{Cos } h_2)}{K\theta} = \frac{EH\ (=R)}{HX\ (=\text{Sin } az_2)}\ .$$

Whence

$$K\theta = \frac{(\text{Sin } az_2)\ (\text{Cos } h_2)}{R\theta} = \frac{47;36,4 \times 58;0,1 \text{ (should be } 58;0,2)}{60}$$

$$= \frac{35,780,974,564 \text{ fourths}}{60} = 46;0,53.$$

(73:13) $WY = OC - K\theta = 5;40,42.$

$WT = \text{Sin } h_1 - \text{Sin } h_2$

(76:5) $= 21;39,54 - 15;21,38 = 6;18,16.$

$TY = (\overline{WY}^2 + \overline{WT}^2)^{\frac{1}{2}} = (417,875,364 \text{ fourths} + 515,108,416 \text{ fourths})^{\frac{1}{2}}$

$= 30,545$ seconds.

In right triangle TYW,

(73;16) $$\frac{TY}{TW} = \frac{\text{Sin } TWY\ (=R)}{\text{Sin } TYW\ (=\text{Cos } \varphi)}\ ,$$

so (76:8) $$\text{Sin } \bar{\varphi} = \frac{TW \cdot R}{TY} = \frac{1,361,760 \text{ seconds}}{30,545 \text{ seconds}} = 44;34,55.$$

The last number is badly rounded, and should have been 44;34,56.

We note that the author could have saved himself the squarings and root extraction involved in calculating TY had he used the tangent function, well known to him, instead of restricting himself to the sine in this last calculation. In any event, he obtains $\varphi = 42;0,35^{\circ}$ for Jurjāniya.

The other six figures illustrating this method in the text are special cases, many of them necessitated by Bīrūnī's lack of negative numbers. The second drawing of Figure 4 is for the situation when both azimuths are south; the third is for one north and one south azimuth; the fourth is for one azimuth zero, the other north; the fifth for one zero the other south. Figure 5 (our C5) is for both azimuths south, one being 90°. Figure 6 is for one azimuth zero, the other 90°, and Figure 7 illustrates the situation if the observer is on the equator.

The case where one of the solar positions is in the meridian is further illustrated by two worked examples, for the φ of Jurjāniya, the observations having been made on the same day as the numerical example just discussed.

The meridian altitude AT (Figure C5) is observed as 24;28°, BM = az_2 as 67;30°, and LM = h_2 as 21;10°. Now

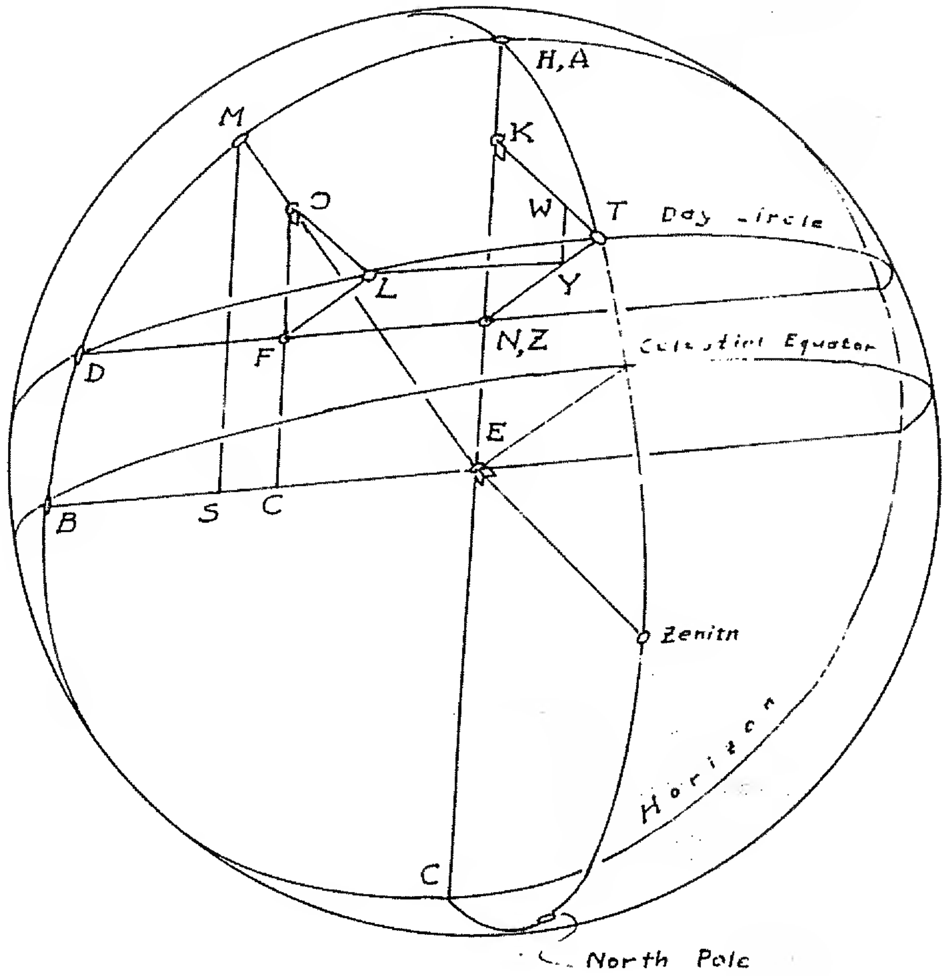


Figure C5

$$(77{:}7)\quad OC = \frac{EO \times MS}{R} = \frac{(\text{Cos } h_2)(\text{Sin } az_2)}{R} = \frac{55;57,7 \times 55;25,58}{60}$$

$$= \frac{40,196,369,266 \text{ fourths}}{216,000 \text{ seconds}} = 51;41,35.$$

(The last digit of the quotient should be 34.)

$$YW = KE - OC = \text{Cos } AT - OC$$
$$= 54;36,44 - 51;41,35 = 2;55,9.$$
$$(77{:}10)\quad TW = TK - LO = \text{Sin } AT - \text{Sin } h_2$$
$$= 24;50,59 - 21;39,53 = 3;11,6.$$

(The value of LO is badly rounded off. The last digit should be 54.)

$$TY = (\overline{TW}^2 + \overline{WY}^2)^{\frac{1}{2}} = (131,469,156 \text{ fourths} + 110,439,081 \text{ fourths})^{\frac{1}{2}}$$
$$= 15,553 \text{ seconds}.$$

$$\text{Cos}\,\varphi = \frac{TW \times R}{TY} = \frac{687,960 \text{ seconds}}{15,553 \text{ seconds}} = 44;13,59.$$

(The quotient is really 44;14,0.)

$$\bar{\varphi} = 47;29,42^\circ,$$

whence the latitude of Jurjāniya is $42;30,18^\circ$.

Use the same figure to let L represent the solar position at the time of a third observation, whereby now $BM = az_3 = 52;30^\circ$, and $LM = h_3 = 14;50^\circ$. So

$$(78;9)\quad OC = \frac{EO \times MS}{R} = \frac{58;0,1 \times 47;36,4}{60} = 46;0,53.$$

(The accurate value of EO is 58;0,1,43.)

$$WY = KE - OC = 54;36,44 - 46;0,53 = 8;35,51.$$
$$(78{:}11)\quad TW = TK - LO = 24;50,59 - 15;21,38$$
$$= 9;29,21.$$

$$TY = (\overline{TW}^2 + \overline{WY}^2)^{\frac{1}{2}} = (1,166,290,801 \text{ fourths} + 957,964,401 \text{ fourths})$$
$$= 460,090 \text{ seconds}.$$

$$\text{Cos}\ \varphi = \frac{TW \times R}{TY} = \frac{2,049,660 \text{ seconds}}{46,090 \text{ seconds}} = 44;28,15.$$

$$\bar{\varphi} = 47;49,56^\circ,$$

whence $\varphi = 42;10,4^\circ$.

After a passing reference to work at the village of Būshkānz during the summer solstice of 994 (see Section 29 below) Bīrūnī

proceeds to describe an additional determination of φ at Jurjāniya, based on observations made on 16 June, 1016, taking one meridian solar altitude (AT in Figure C6) and one when the sun was due east (at L).

He found AT to be $71;18^\circ$, and BL $36;30^\circ$. Now

$$(80{:}6)\quad TW = TK - LO = \text{Sin } TA - \text{Sin } BL = 56;49,57 - 35;41,22$$
$$= 21;8,35.$$

$$YW = EK = \text{Cos } AT = 19;14,12.$$

$$TY = (\overline{TW}^2 + \overline{YW}^2)^{\frac{1}{2}} = (5,973,493,225 \text{ fourths} + 4,795,839,504 \text{ fourths})^{\frac{1}{2}}$$
$$= 102,904 \text{ seconds}.$$

$$\text{Sin}\ \varphi = \frac{(\text{Cos } AT) \times R}{TY} = \frac{19;14,12 \times 60}{102,904 \text{ seconds}} = \frac{4,155,120 \text{ seconds}}{102,904 \text{ seconds}}$$

$$= 40;22,43, \text{ whence}$$
$$\varphi = 42;17,50^\circ.$$

Bīrūnī here (81:1) expresses his distrust of results arrived at by lengthy trigonometric computations. In point of fact, most of his operations are precise to three significant sexagesimal digits, i.e. one part in 216,000, and his maximum error is one unit in the third place.

He prefers methods which permit direct inference of the result, and by way of illustration gives the derivation of the value he prefers for the φ of Jurjāniya. It is

$$(81{:}7)\quad \bar{\varphi} = h - \varepsilon = 71;18^\circ - 23;35^\circ = 47;43^\circ,$$

whence $\varphi = 42;17^\circ$,

where here h is the summer solsticial solar altitude at the locality.

We collect here the other four determinations of the same parameter:

(76:10)	$42;0,35^\circ$
(78:5)	42;30,18
(78:16)	42;10,4
(80:14)	42;17,50

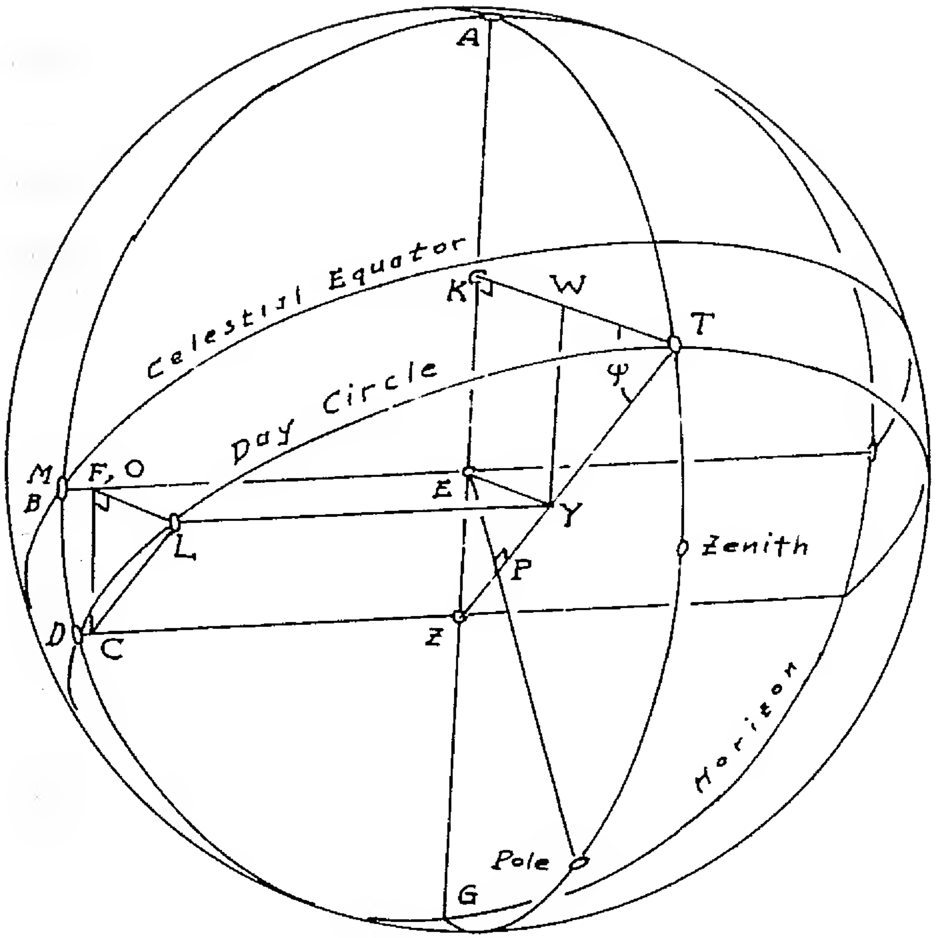


Figure C6

The mean of this set rounds off to 42;15° which is close to but not identical with Bīrūnī's adopted value.

The section closes with a discussion of what happens when the locality is on the equator. Then, referring again to Figure C4, since any day-circle is normal to the horizon and $\varphi = 0$, the two triangles OLF and KTN (81:15) shrink into the lines TN and LF respectively. The first steps in the computation are the determination of the "shares of the azimuths", OC and Kθ. It is easy to see that whenever $\varphi = 0$ these two segments will be equal. Whenever this happens the observer should realize that he is on the equator.

Jurjāniya was one of the two major cities of medieval Khwārazm. Its site is on the left bank of the Oxus, about six kilometers from the river, on the southern edge of modern Kunya-Urgench. Its latitude is 42;18°. (See LeStr., p. 446 ff.; RT, p. 297, note 339.)

We note the mention in this passage of two observational instruments, both large and presumably fixed: at Būshkānz (79:4) a horizontal circle of diameter fifteen cubits (8.1 m., cf. Hinz, p. 55), at Jurjāniya a quadrant of diameter six cubits (c. 3.2m.).

18. Difference in the Latitudes of Two Localities from Meridian Altitudes, Three Examples (82:10 – 87:14)

It is sufficiently clear without a picture that if the meridian altitude of the same fixed star is observed from two different localities, the difference in the altitudes will equal the difference in the latitudes of the two places. This is the essence of the present section, but as usual Bīrūnī is constrained by his lack of negative numbers to state all manner of special rules for special cases. He remarks that the same method may be used by observations on the sun so long as they are made at a time in the one locality when the sun has the same declination as it had when observed at the other. By the same token, if the time between observations on a fixed star is so great that precessional changes in its coordinates cannot be neglected, then elaborate trigonometric transformations must be used.

The first of the examples given in the text utilizes the star UMa ζ (66:13, Section 15 above), the two localities being Baghdad and Sāmarrā. The latter place is on the Tigris, about twenty-five miles upstream from Baghdad. It was the ʿAbbasid capital between 836 and 892 (LeStr., p. 53). Both sets of observations were

carried out by the Banū Mūsā, presumably c.863. Bīrūnī accepts an independently determined latitude of 34;12° for Sāmarrā (85:12, the modern value is 34;13°). Then using both variants already reported for the Baghdad observation,

Culmination at Sāmarrā	63;5°	63;5°
Culmination at Baghdad	62;13	62;3
Difference	0;52	1;2
Latitude of Sāmarrā	34;12°	34;12°
Difference	0;52	1;2
Latitude of Baghdad	33;20	33;10

On the basis of this result Bīrūnī discards the second reading (86:3).

The second example involves meridian transits of the sun at Baghdad and Damascus, where the latitude of the latter is taken to be 33;30,18°, derived in the Taḥdīd, 86:11, see Section 21 below. (The modern value is 33;30°.) The Damascus observations were those directed by Khālid al-Marwarūdī and discussed later in the Taḥdīd (90:15). It is difficult to identify the Abū al-Ḥasan (86:9, 15) who supplied the Baghdad results. Bulgakov (RT, p. 300, note 385) suggests the astrologer Abū al-Ḥasan al-Ahwāzī, who is known to have lived in Baghdad at this time.

The columns below are for Wednesday, 1 May, 832, and Saturday, 3 August, 832, respectively.

Culmination at Baghdad	72;14°	73;7°
Culmination at Damascus	72;7,50	73;2,4
Difference	0;5,10	0;4,56
Latitude of Damascus	33;30,18°	33;30,18°
Difference	0;5,10	0;4,56
Latitude of Baghdad	33;24,8	33;25,22

For the third example, one of the localities is Rayy, an ancient city of central Iran, the site of which is at the present village of Shāh ʿAbd al-ʿAẓīm, just south of Tehran. The latitude of Rayy is taken to be 35;34,39° (modern value 35;35°). The other locality is Būshkānz, not named in this passage (87:3), but clearly the place referred to in 79:1, since the time of the observation is the summer solstice of 994 (see Section 17 above). The observer at Rayy, al-Khujandī, reappears later in the Taḥdīd (see Section 26 below).

Culmination at Rayy	77;57,40°
Culmination at Būshkānz	71;59,45
Difference	5;57,55
Latitude of Rayy	35;34,39°
Difference	5;57,55
Latitude of Būshkānz	41;32,34

If, on the other hand the independently determined latitude of Būshkānz (79:8) of 41;36° is accepted then

Latitude of Būshkānz	41;36°
Difference	5;57,55
Latitude of Rayy	35;38,5

19. Ptolemy's Determination (88:1 – 89:21)

The meridian solar altitude at the summer solstice is $\bar{\varphi} + \varepsilon$; at the winter solstice $\bar{\varphi} - \varepsilon$. Hence, as Bīrūnī says, ε is half the difference between them. What he means by the "second method" (88:8) of determining ε is not clear.

His information on Ptolemy and the latter's predecessors is accurately transmitted from Almagest I, 12 (vol. I, p. 44). To express Eratosthenes' parameter sexagesimally in degrees we have

$$2\varepsilon / 360^{\circ} = 11/83;$$

whence

$2\varepsilon = 11 \times 360^{\circ}/83 = 47;42,39,2,10,7,13,44^{\circ}$. Bīrūnī has (89:4) 47;42,39,2,10,7,14,13, his last two digits being erroneous. ε should be

23;51,19,31,5,3,36,52,3, instead of Bīrūnī's
23;51,19,31,5,3,37,6,30.

He writes that since Ptolemy's findings were confined within the limits shown below,

$$47;40^{\circ} < 2\varepsilon < 47;45^{\circ},$$

he took the midpoint of the spread, $2\varepsilon = 47;42,30$, whence $\varepsilon = 23;51,15$. In fact the last two values are not mentioned by Ptolemy. He only says that his results bear out the earlier findings of 23;51,20°, which he proceeds to adopt for the declination tables (Almagest, vol. I, p. 54).

Nevertheless, it is of interest that, at least by the eleventh century, it was accepted that the most favorable value for a set of determinations of a single quantity was the arithmetic mean between the extremes of the spread.

20. Yaḥyā and Khālid at Baghdad and Damascus (83:22 – 91:12)

The text now describes observations made at the behest of the Abbasid caliph al-Ma'mūn. In charge at first was Yaḥyā b. abī Manṣūr, a sometime Mazdean from northern Iran (Suter, p. 8; K&F). Ma'mūn (reigned 813-833) was a son of the famous Harūn al-Rashīd. In his time the dynasty still enjoyed great power.

Shammāsiya was a suburb of Baghdad located northeast of the city proper and on the east bank of the Tigris (LeStr., p. 31). The operations there carried out during the year 828 yielded the result

$$\varepsilon = \frac{h_{max} - h_{min}}{2} = \frac{(\bar{\varphi} + \varepsilon) - (\bar{\varphi} - \varepsilon)}{2} = \frac{79;6^{\circ} + 32;0^{\circ}}{2} = 23;33^{\circ}$$

They were observed by Muḥammad b. Mūsā al-Khwārizmī (fl. 820, Suter, p. 10), author of the famous algebra and zīj.

For the following year the results give

$$\varepsilon = \frac{80;8^{\circ} + 32;58^{\circ}}{2} = 23;35^{\circ}$$

The work at Damascus was facilitated by constructing a mural quadrant with a dimension (radius?) of about 5.4 meters (cf. Hinz, p. 55). This installation inaugurated the trend among Muslim astronomers toward building ever larger instruments in order to attain better precision (cf. Sayili). The director of the new observatory was Khālid b. ʿAbd al-Malik, concerning whom we have no other information (Suter, p. 11). From 91:2 and 86:7 it seems clear that observations were made and recorded daily, and that Bīrūnī had at hand a copy of the observation records.

Those used here would seem to be

(1) The winter solstice of 17 Dec., 831, $h_{min} = 32;56^{\circ}$.
(2) The summer solstice of 17 June, 832, $h_{max} = 80;3,55^{\circ}$.
(3) The winter solstice of 16 Dec., 833, $h_{min} = 32;55^{\circ}$.

From (1) and (3)

$$\varepsilon = \frac{80;3,55^{\circ} - 32;56^{\circ}}{2} = 23;33,57,30^{\circ},$$

and from (2) and (3)

$$\doteq \frac{80;3\ 55^{\circ} - 32;55^{\circ}}{2} = 23;34,27,30^{\circ}.$$

The second of these results is discarded because the observations stretched over more than a year. The first is chosen, with the terminal digit suppressed. Bīrūnī notes the scribal error of 52 for 57 (in 91;10) in the report of Sanad b. ʿAlī, another prominent Abbasid astronomer (Suter, p. 13).

The data from Damascus are now subjected to the elaborate treatment discussed below.

21. Reduction of the Damascus Observations (91:13 – 94:10)

On the basis of meridian solar altitudes observed on three successive days, Sunday, Monday, and Tuesday (16, 17, and 18 respectively, of June, 832), Bīrūnī proposes to calculate the time of the summer solstice that year and the maximum solar altitude i.e. $\bar{\varphi} + \varepsilon$ for Damascus.

Before proceeding to explain what he does, however, it is necessary to clear up an inconsistency in the dates he gives. The Hijra date 10 Jumādā I, 217 does not fall on a Monday in either the popular or the stronomical reckoning. The Monday for the week-day of the middle observation is, however, secure, for the text names Sunday and Tuesday on either side. The Yazdigerd date, 22 Urdībihisht, 201, converts into 17 June 832, a Monday. Its Hijra (popular) equivalent is 14 Jumāda I 214. If we restore to the text (19:15) the word al-rābaʿa preceding the ʿashar, thus making the ten into the fourteenth, all will be well.

The main drift of Abū Rayḥān's argument can be seen by a glance at Figure C9, which is a schematic adaptation of the text's Figure 9. The horizontal axis represents the ecliptic, the points A, B, and G giving the solar true longitudes at these points, at noon on Sunday, Monday and Tuesday respectively. The ordinates show meridian altitudes. We seek the maximum possible meridian altitude. This, in general, will not be observed for it occurs at the summer solstice, and in general the instant of the solstice will not coincide with local apparent noon. For reasons of symmetry, as Bīrūnī points out (92:5), the solstice did not occur at noon of Monday. He proceeds, by an admittedly approximate method (92:13) to estimate the time of the solstice, and the solar altitude had noon coincided with it. To do this he assumes that the declination is a linear zigzag function, to which $\bar{\varphi}$ may be added to produce the broken line on Figure C9. To do this, it suffices to pass a line through h_A and h_B; pass a second line through h_G with slope the negative of the first line, and to find the intersection of the two lines. We note that (91:18):

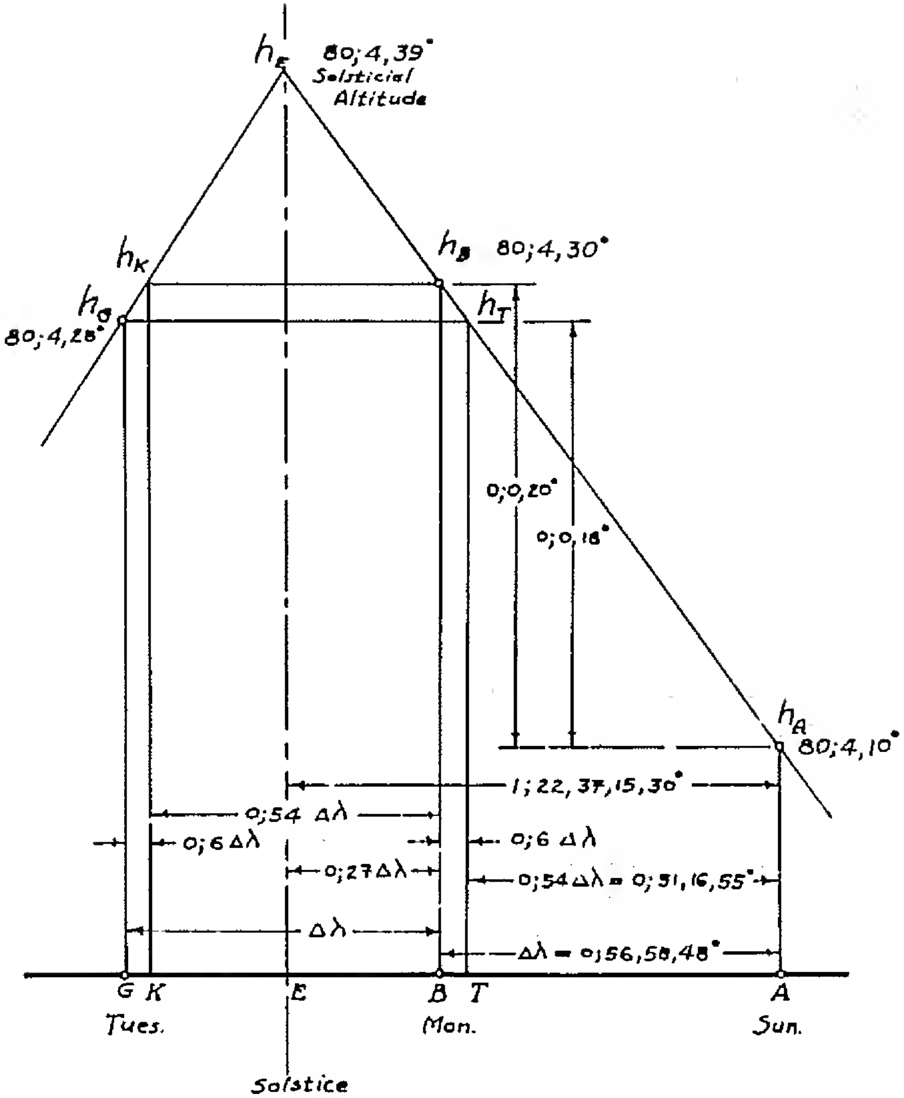


Figure C9

$$h_A = 80;4,10^\circ$$
$$h_B = 80;4,30^\circ$$
$$h_G = 80;4,28^\circ$$

The author states (92:17) that $AB = 0;56,58,48^\circ$ ($= \Delta\lambda$). There was no need to calculate this at all, since we are interested only in the ratios of segments along AG. Nevertheless it is of some interest to verify this number. We cannot hope for complete success, since we do not know what zīj Bīrūnī based his results on. His Masudic Canon was not yet finished, but from frequent references to al-Battānī's zīj in the Taḥdīd, we know he had access to it, and that he thought highly of it. We seek

$$\Delta\lambda_{AB} = \lambda_B - \lambda_A = (\bar\lambda_B - e_B) - (\bar\lambda_A - e_A) = \Delta\bar\lambda_{AB} - \Delta e_{AB}$$

$\Delta\bar\lambda_{AB}$ is simply the solar mean travel in a day, $0;59,8,21^\circ$. To find the Δe_{AB} we use Battānī's solar equation table (vol. II, p. 78) to obtain

$$e(8^\circ) = 0;16,1^\circ$$
$$e(7^\circ) = 0;14,1^\circ,$$

the argument of eight degrees being from 92:17. So

$$\Delta e_{AB} = 0;59,8,21^\circ/d \times (e(8^\circ) - e(7^\circ)) = 0;59,8,21 \times 0;2,0$$
$$= 0;1,58,17,$$

and $\Delta\lambda_{AB} = 0;59,8,21 - 0;1,58,17 = 0;57,10,4^\circ$,

which is not very close to the text's $0;56,58,48$.

Now

$$AT = \frac{h_T - h_A}{h_B - h_A} AB = \frac{0;0,18}{0;0;20}(0;56,58,48) = 0;51,16,55. \quad (92:18)$$

And

$$AE = (BK/2) + AB = (AT/2) + AB = 0;25,38,27,30 + 0;56,58,48$$
$$= 1;22,37,15,30^\circ \quad (92:21)$$

Also

$$\frac{h_E - h_A}{h_B - h_A} = \frac{AE}{AB} = \left(\frac{1;27\ \Delta\lambda}{1\Delta\lambda}\right). \quad (92:22)$$

So

$$h_E - h_A = 1;27(h_B - h_A) = 1;27(0;0,20) = 0;0,29^\circ.$$

Hence the solsticial altitude is $h_E = 80;4,39^\circ$ (93;3).

To one accustomed to the use of smooth functions, Bīrūnī's use of linear methods in the vicinity of an extremal seems strange, especially since he was well aware of the fact that the rate of change in the declination is small in the vicinity of its maximum. Dr. Churchill Eisenhart, by passing a parabola with vertical axis through points h_A, h_B, and h_G, obtained

$$AE = 1;21,32^\circ,$$

and

$$h_E = 80;4,32^\circ.$$

In the same way Bīrūnī calculates what the meridian solar altitude would have been had the winter solstice occurred at noon. He uses the following three observations (93:7, see Figure C10):

Monday 15 December, 832,	$h_A = 32;55,0^\circ$
Tuesday 16 December, 832,	$h_B = 32;54,58^\circ$
Wednesday 17 December, 832,	$h_G = 32;55,28^\circ$

This time $\Delta\lambda = BG$ is said to be $1;1,27,36^\circ$. The sun is now near perigee and at nearly its maximum angular velocity. Bīrūnī calculates

$$TG = \frac{h_G - h_T}{h_G - h_B}\Delta\lambda = \frac{0;0,28}{0;0,30}\ 1;1,27,36 = 0;57,21,47,28^\circ,$$

but the text has (93:16) $0;57,21,46^\circ$.

Then (94:1)

$$GE = (TG/2) + 1;1,27,36 = 1;30,8,28^\circ,$$

and since $EB = TG/2$, $\Delta h_{EB} = \frac{1}{2}\Delta h_{TG} = 0;0,14^\circ$.

Hence $h_E = h_G - \Delta h_{EG} = h_G - (\Delta h_{EB} + \Delta h_{BG}) = 32;55,28 - (0;0,14 + 0;0,$
$= 32;54,44^\circ$.

Now

(94:10) $$\varepsilon = \frac{h_{max} - h_{min}}{2} = \frac{80;4,39 - 32;54,44}{2} = 23;34,57,30^\circ$$

Bīrūnī points out (94:10) that if the observation results are used directly without any interpolation one obtains

$$\varepsilon = 23;34,51^\circ.$$

Perhaps this is as good a place as any to discuss the precision of these and related observations. It has been estimated (by Aaboe & Price, pp. 2-9) that for man-sized pre-telescopic instruments the accuracy of angular measurements can hardly be better than $0;5^\circ$.

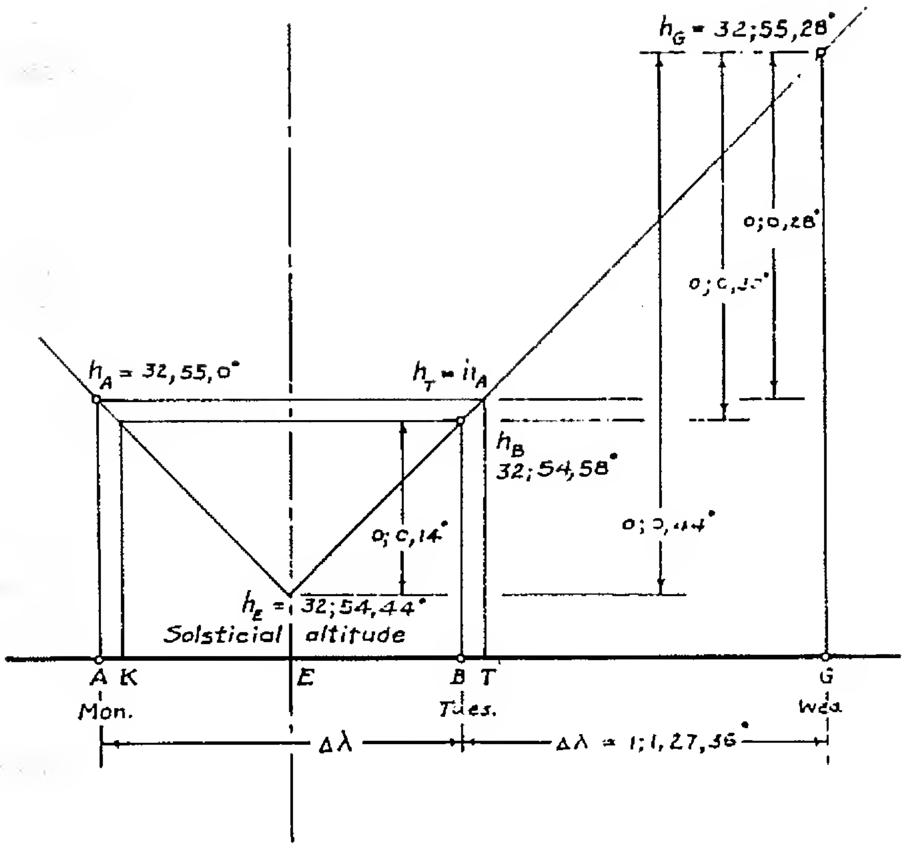


Figure C10

Of course, the Damascus quadrant was much larger than a man. Nevertheless, it is quite certain that for any non-telescopic observation the maximum precision attainable is of the order of a minute of arc. Hence any observation purporting to give seconds is misleading to this extent.

A latitude for Damascus is readily obtainable from the data as follows:

$$\varphi = 90^\circ - \frac{h_{max} + h_{min}}{2} = 90^\circ - \frac{80;4,39^\circ + 32;54,44^\circ}{2}$$

$$= 33;30,18,30^\circ .$$

Although this result is not explicitly derived in the text, it has already been used (in 86:11), truncated to seconds.

22. Observations at Samarra, Baghdad, and Raqqa (94:11 – 96:2)

The Banū Mūsā (see Section 15 above), observing from Sāmarrā, found for:

The summer solstice of 17 June, 857, $h_{max} = 79;22^\circ$.
The winter solstice of 16 Dec., 857, $h_{min} = 32;13^\circ$.

Hence

$$\varepsilon = \frac{79;22 - 32;13}{2} = 23;34^\circ .$$

The same h_{min} was obtained for the winter solstice two years later. There is a discrepancy in the dates cited. The seventeenth of Ramaḍān, 245, by the popular reckoning was 16 Dec., 859, a Saturday, whereas Sunday is cited (95:1). By the popular reckoning it would be a Friday, which is worse. But the first epagomenal day of 228 Yazd. was Sunday, 17 Dec., 859, and this is probably the day intended.

The same astronomers, observing at Baghdad (95:5), obtained for

The winter solstice of 16 Dec., 868, $h_{min} = 33;5^\circ$.
The summer solstice of 17 June, 869, $h_{max} = 80;15^\circ$.

Hence

$$\varepsilon = \frac{80;15 - 33;5}{2} = 23;35^\circ .$$

One of the sources of the above information was al-Nairīzī (d.c. 920, Suter, p. 45), a well-known mathematician and astronomer. The other was al-Khāzin (see Section 12 above). Their Almagest commentaries are not extant.

Next given are the results obtained by al-Battānī (d. 929) an astronomer justly famed in the Orient, but also in Europe where he was known as Albategnius (Suter, pp. 45–47). The observations cited here spanned the years 880 to 884 and were made at Raqqa, a city on the upper Euphrates (LeStr., p. 101), before he moved to Baghdad. He finds

$$\varepsilon = \frac{59;36^o - 12;26^o}{2} = 23;35^o \qquad (95:17)$$

23. Observations at Balkh, Marv, and Rayy (96:3 – 99:4)

The work of Sulaimān b. ᶜIṣma is cited frequently by Bīrūnī, but hardly ever by others (cf. Suter, p. 56, where his name is incorrectly given as ᶜOqba). For Balkh, the site of his observatory, see Section 8 above. His results are for

The winter solstice of 14 Dec., 888, $h_{min} = 29;46^o$ corrected to $29;47,17,6^o$.
The summer solstice of 17 June, 889, $h_{max} = 76;54^o$ corrected to $76;54,41,23^o$.

In the text at 96:10 the Hijra version of the date is given wrong. Replace "Muḥarram" by "Ṣafar".

The uncorrected altitudes lead to

$$\varepsilon = \frac{76;54^o - 29;46^o}{2} = 23;34^o,$$

while the corrected versions give

$$\varepsilon = \frac{76;54,41,23^o - 29;47,17,6^o}{2} = 23;33,42,8,30^o.$$

The dimension of the instrument given as eight cubits (c. 4.3 meters, see Hinz, p. 55) is translated as diameter. The text has quṭr, which is sometimes used also for diagonal. What it means in the case of a quadrant is difficult to say.

In 820 a certain Ṭāhir was appointed governor of Khurāsān by the caliph al-Ma' mūn. He established a semi-independent dynasty,

of which our Manṣūr b. Ṭalḥa (96:17, L.-P., p. 128) was the last member. The work by him named in 97:8 is not extant. The Tahirid capital was at Marv (or Merv, modern Mary), a great and ancient city located east of the southern end of the Caspian and south of the Aral. Observations there, of unknown date, yielded

$$\varepsilon = \frac{h_{max} - h_{min}}{2} = \frac{75;52^o - 28;46^o}{2} = 23;33^o. \qquad (97:14)$$

We have no information concerning Muḥammad b. ᶜAlī al-Makkī (fl. 850) beyond what is given here and in four other passages in the Taḥdīd. He carried out observations at Nīshāpūr in Khurāsān. Neither of al-Makkī's books mentioned in the Taḥdīd are extant. Apparently he was partial to Indian science (112:5).

The region defined by a "climate" does not correspond to a single latitude, but to a range of latitudes (see Section 24 below). Hence the $35;26^o$ given at 98:4 is not to be taken seriously.

Background on the Buwayhid official, Ibn al-ᶜAmīd, has been given in Section 9 above. The Buwayhids originated in the Caspian province called Dailam, hence the reference to "the Dailamite state" in 98:7 in connection with the mural quadrant at Rayy.

Concerning Abū al-Faḍl al-Hirawī, most of the information available comes from the Taḥdīd. In 167:4 and 212:11 Bīrūnī names and quotes from a book of his, simultaneously voicing a high opinion of him. Al-Hirawī was also the author of a recension of Menelaos' Spherics. From 212:11 we infer that he worked in Rayy in the middle of the tenth century. The passage at 245:2 implies that in March of 982 and 983 he was making solar observations at Jurjān. (Suter, p. 228; see also Section 52 below).

Al-Khāzin has been encountered twice previously (Sections 12 and 22).

To obtain the maximum solar altitude, meridian transits were observed on each of five successive days, with the results shown below:

	h
Wednesday 22 June 959	$78;3^o$
Thursday	78;5 less a little
Friday	78;6
Saturday	78;6 less a little
Sunday	76;5

There is a discrepancy in the date given for the winter solstice. Friday 21 Shawwal, 349 is 13 December, 960, astronomical reckoning. This is not the vicinity of the winter solstice following the spring

solstice above, but the one a year later. On the other hand, 19 Ādhar, 328 Yazd. (old style) is 15 December, 959, the proper year. Probably this is the solstice observed. The results are

	h
Friday	$30;47^\circ$
the following Sunday	30;46 plus a little

Hence

$$\varepsilon = \frac{h_{max} - h_{min}}{2} = \frac{78;6^\circ - 30;46^\circ}{2} = 23;40^\circ$$

Also

$$\varphi = 90^\circ - \bar{\varphi} = 90^\circ - \frac{h_{max} + h_{min}}{2} = 35;34^\circ \qquad (99:1)$$

24. Observations at Shīrāz and Baghdad (99:5 – 100:16)

ᶜAḍud al-Dawla abū Shujāᶜ Khusro was from 949 to 982 the Buwayhid ruler of Fārs, the southwestern province of Iran, Shīrāz being its capital (L.-P., p. 141). The meridian ring set up there was about 1.4 meters in diameter (Hinz, p. 55). In charge of the operation was ᶜAbd al-Raḥmān al-Ṣūfī, author of the famous star catalogue (Suter, p. 62). Among the witnesses were:

Al-Kūhī, best known as a geometer (Suter, p. 75).
Al-Sijzī, also primarily a geometer (Suter, p. 80).
Naẓīf b. Yumn the Greek, a physician and translator of Greek scientific works into Arabic (Suter, p. 68).
ᶜAbdallah, Ghulām Zuḥal, an astrologer (Suter, p. 63). His nickname, "the servant of Venus", is strange for a Muslim.

Meridian transits for the winter solstice of 969 were observed on three successive days and the result for the same solstice the following year were:

		h
Wednesday,	15 December, 969	$36;50^\circ$
Thursday,		36;49
Friday,		36;50
	16 December, 970	36;49

For the summer solstice, transits on three successive days showed

	h
Thursday, 16 June, 970	83;59 less a little
Friday	83;59
Saturday	83;59 less a little.

Hence

$$\varepsilon = \frac{h_{max} - h_{min}}{2} = \frac{83;59^\circ - 36;49^\circ}{2} = 23;35^\circ \qquad (100:4)$$

Abū al-Wafā' al-Būzjānī was a very able mathematician and astronomer from Khurāsān (Suter, p. 71). His observations at Baghdad during 976 and 977 or thereabouts (100:8) led to the same value for ε given above. Only a fragment of his Almagest is extant. ᶜIzz al-Dawla Bakhtiyār was another Buwayhid prince (L.-P., p. 141).

In c. 985, working also at Baghdad, al-Ṣaghānī obtained the same ε as al-Ṣūfī and Abū al-Wafā'. Al-Ṣaghānī's book Qawānīn is not extant, although others of his writings are (Suter, p. 65). Assuming, on the basis of 99:6, that one span (shibr) is half a cubit, the diameter of his meridian circle was about 1.6 meters (Hinz, p. 55).

25. Al-Kūhī at Baghdād (100:17 – 101:19)

In 982 Sharaf al-Dawla succeeded his father ᶜAḍud al-Dawla, whose sway had by then been extended over ᶜIrāq (L.-P., pp. 141-144). Under his patronage al-Kūhī (see Section 24 above) constructed at Bāghdād the interior surface of a spherical segment of diameter c. 13.5 meters and having an aperture at its center. Thus the daily motion of the sun in the celestial sphere was visibly reproduced.

With this instrument the meridian solar altitude for the summer solstice of 16 June, 988, was observed. Bīrūnī is right to ridicule the notion that from this Ptolemy's inaccurate value of ε could be derived. Rather the Ptolemaic value was assumed and then the latitude of Baghdad derived.

The "rotation of the ecliptic poles" (101:11) is a reference to the theory of the trepidation of the equinoxes. The latter was expounded by Thābit b. Qurra (see Section 11 above) in a treatise which has survived in Latin translation and which has been put into English and explained (Thābit). At 101:12 the text has Ibrāhīm b. Sinān (not just Sinān), who was a grandson of Thābit. Evidently he also wrote on trepidation, but the book has disappeared. Al-Khāzin (Section 12) also discussed trepidation, in his Zīj al-Ṣafā'iḥ, which likewise is not extant (see Chron., transl., p. 322).

26. The Fakhrī Sextant at Rayy (101:20 – 102:10)

Still another Buwayhid prince sponsored scientific research in these times. Fakhr al-Dawla, brother of ʿAḍud al-Dawla ruled central Iran with Rayy as his capital from 976 to 997 (L.-P., pp. 142-144). He commissioned a certain al-Khujandī to build a mural instrument far more massive than any seen up to that time, a sextant of diameter c. 43 meters. It is described in our text, and there are other notices of it in the literature (e.g. Sédillot, p. 202; Sayili, p. 118).

Al-Khujandī is best known for the sextant he constructed, the prototype of even larger instruments, but he also wrote on geometry, number theory, and a certain type of astrolabe (Suter, p. 74).

27. Reduction of al-Khujandī's Observations (102:11 – 107:6)

The data consist of two meridian transit altitudes each for the summer and winter solstices of the year 994. These four numbers are shown displayed in Figure C11 as h_A, h_B, h_D, and h_H. Since there are only two per phenomenon, in contrast to three each for the Damascus observations (91:14), al-Khujandī is unable to deduce the altitudes for the two solstices. Instead he has recourse to a trigonometric approach from which ε is obtained. Since $h_A = h_B$, he infers from considerations of symmetry that the summer solstice occurred at midnight of Saturday, 16 June. The two horizontal lines AB and TD on Figure C11 represent opposite segments on the ecliptic. Using al-Battānī's Zīj, Al-Khujandī calculated that AG and GB, each representing half a day's solar travel in this part of the ecliptic, are each $0;28,36^{\circ}$. This compares favorably with our results of $0;28,35^{\circ}$ $(=0;57,9 / 2)$ obtained by interpolation in the same zīj (cf. the commentary to 92:17).

The situation with the winter solstice is more involved because of the asymmetry in the observations. Al-Khujandī apparently decided to ignore here the slight difference between h_H and h_D, and, for part of the derivation which follows, acted as though midnight of Saturday, 15 December, were the instant of the solstice, the point K midway between the two observations. He calculates that $HK = 1;31,48^{\circ}$, corresponding to a daily solar motion in this part of the ecliptic of $1;1,12^{\circ}$. Bīrūnī points out that since $h_H < h_D$, the solstice must have occurred nearer to noon on Monday than noon

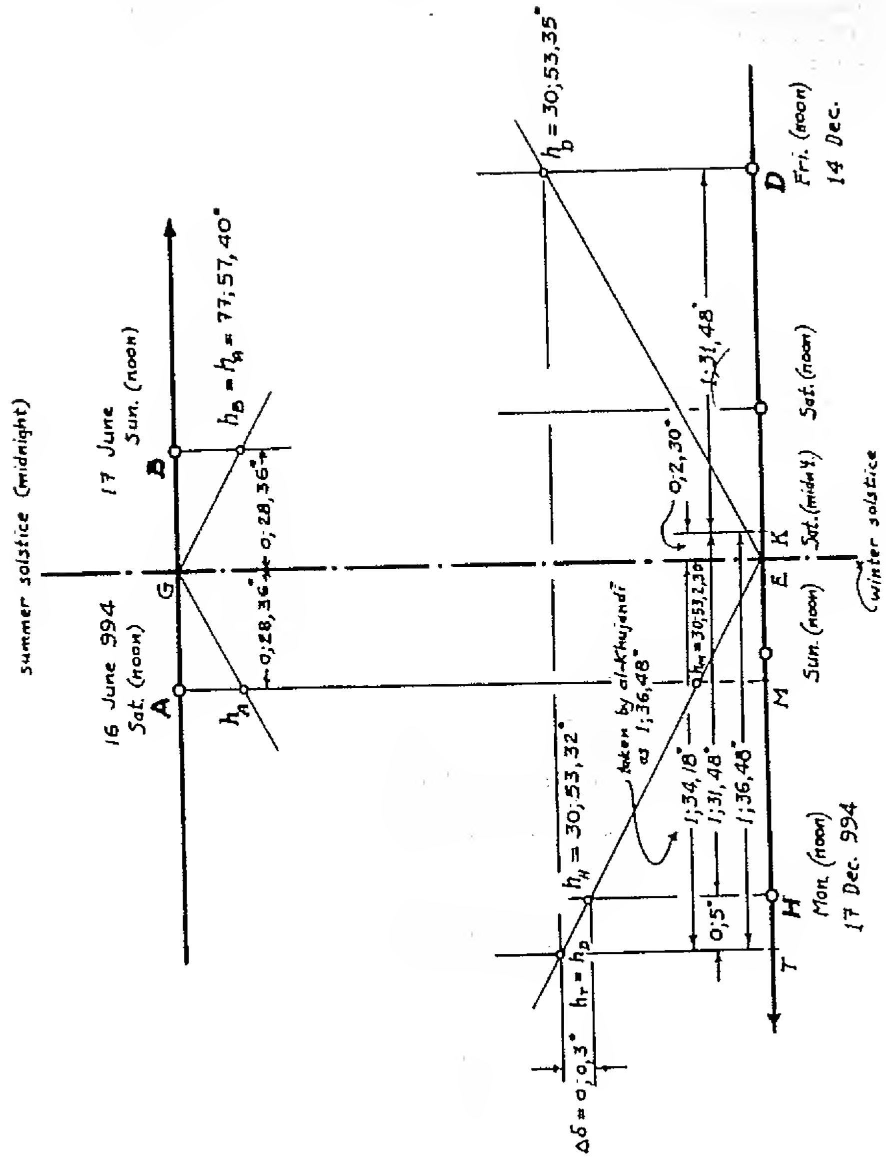


Figure C11

on Friday, and he proceeds to calculate the displacement, EK, the ecliptic distance between midnight Saturday (K) and the winter solsticial point (E). Mark the point T in such manner that TE = ED. Then, by considerations of symmetry, $h_T = h_D = 30;53,35^o$. The difference in declination between the sun when it is at T and when it is at H equals the difference in its meridian altitude at these positions, i.e.

$$\Delta\delta = h_T - h_D = 30;53,35 - 30;53,32^o = 0;0,3^o.$$

From this it is inferred that the corresponding $\Delta\lambda$, i.e. TH, is $0;5^o$. In principle, since the position of E is yet to be determined, the argument of the declination function cannot be located on the ecliptic, and $\Delta\lambda$ cannot be determined. Probably al-Khujandī fixed the argument of $\delta(\theta)$ by putting $\theta = 90^o$ at K. We will see that the error involved cannot be large. Since EK = TH/2, if TH = $0;5^o$, EK must be $0;2,30^o$. Then TE = $1;34,18^o$ (although Bīrūnī states that al-Khujandī uses $1;36,48^o$ for it), and HE = $1;31,48^o$.

We are now in a position to examine the determination of TH (= $\Delta\delta$) by working backwards from it to $\Delta\delta$. Using the declination table in the <u>Canon</u> (vol. I, p. 377, tabulated to four sexagesimal places for each integer degree), and linear interpolation we have

$$\begin{aligned}\Delta\delta &= \delta(\overline{1;34,18^o}) - \delta(\overline{1;34,18^o + 0;5^o})\\ &= \delta(88;26^o) - \delta(88;21^o)\\ &= 0;0,3,25^o,\end{aligned}$$

which rounds off to the text's $0;0,3^o$ (104:5).

Having found the winter solsticial point, the next step is to draw AM parallel to GE meeting TD in M, and to calculate the difference in declination between T and M. Using the same declination tables as above, and al-Khujandī's wrong value of $1;36,48$ for ET, we have

$$\begin{aligned}\Delta\delta &= \delta(M) - \delta(T) = \delta(\overline{0;28,36^o}) - \delta(\overline{1;36,48^o})\\ &= \delta(89;31^o) - \delta(88;23^o) = 23;34,53^o - 23;34,21^o\\ &= 0;0,32^o.\end{aligned}$$

which is as precise as the tables and linear interpolation will permit, and which, to this extent, confirms the text's $0;0,32,30^o$ (105:5).

Of course, the use of a declination table in principle begs the whole question, since an essential parameter of the latter is ε, and it is precisely the determination of ε which is the object of the whole procedure. But in fact a new value of ε will not differ much

from the old, hence it is legitimate to use any declination table for $\Delta\delta$ over a short span.

Now, since the difference in meridian altitudes is the difference in declinations,

$$\begin{aligned}h_M &= h_D - \Delta\delta = 30;53,35^o - 0;0,32,30^o\\ &= 30;53,2,30^o.\end{aligned}$$

Also, the difference in meridian elevation between A and M is

$$\begin{aligned}h_A - h_M &= 77;57,40^o - 30;53,2,30^o\\ &= 47;4,37,30^o, \qquad (105:9).\end{aligned}$$

Attention is now transferred to the configuration on the celestial sphere portrayed in Figure C12. Represented on it are two positions of the ecliptic, one, HKZ, at the time when the summer solsticial point (H) crosses the meridian, the other, GLZ, its position at meridian transit of the winter solsticial point. Points K and L on the new figure correspond to A and M on Figure C11. Hence HK = GL = $0;28,36^o$, and, since AK and ML are parallels of latitude, the arc AM equals the difference between the meridian altitudes on the earlier figure, hence its length is $47;4,37,30^o$. The parameter E can now be calculated in terms of these two numbers. By similar triangles

(106:19) ES (=Cos KH) / ST (=Crd AM) = EH (=Sin 90^o) / GH (=Crd 2ε).

Whence

$$\begin{aligned}\text{Crd } 2\varepsilon &= (\text{Crd AM}) \times R / \text{Cos KH} = 47;55,26 \times 60 / 59;59,53\\ &= 47;55,31,35,\end{aligned}$$

or

(107:5) $$\begin{aligned}\text{Sin } \varepsilon &= 23;57,45,48,\\ \varepsilon &= 23;32,21^o,\end{aligned}$$

where, as usual, the radius of the sphere, R = 60. All these calculations are accurate as shown in the text, save for the last digit of ε. The accurate result is $23;32,22^o$.

As it happens, the source from which Bīrūnī obtained his information (<u>Al-Khujandī</u> in the bibliography) is extant and has been published; a German translation may be found in <u>Schirmer</u>, pp. 63–79. Bīrūnī has faithfully transmitted all of al-Khujandī's numbers, and his Figure 12 is an adaptation of a figure in the source. Al-Khujandī goes on to calculate the latitude of Rayy. He then compares other findings of ε with his own and concludes that this parameter is subject to a very slow periodic secular variation.

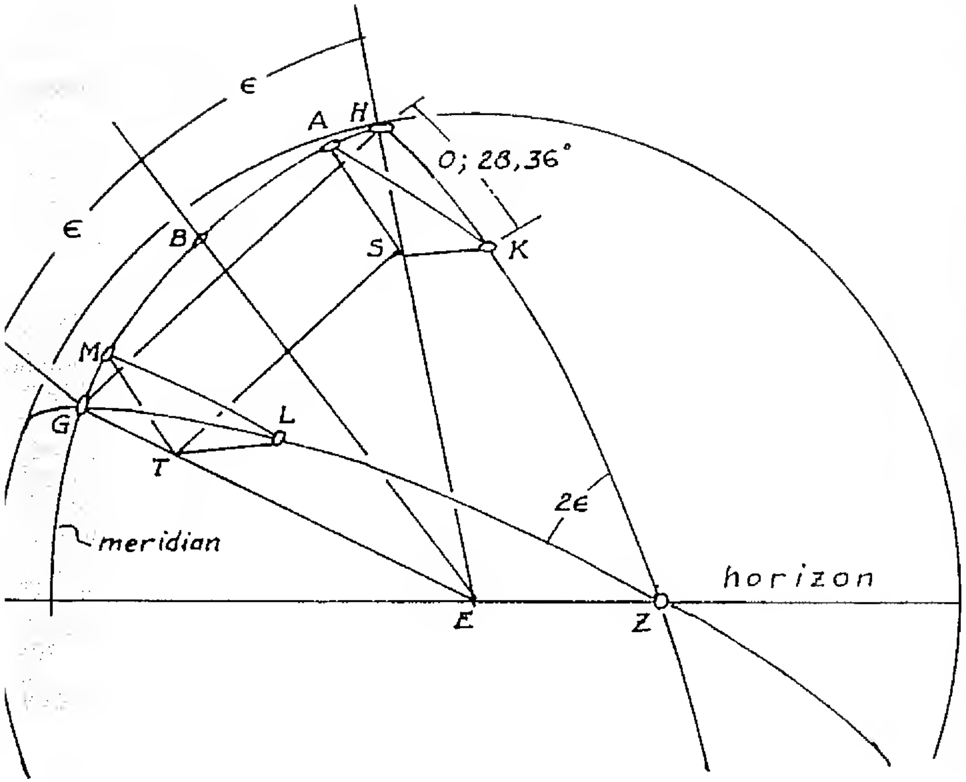


Figure C12

28. Bīrūnī's Discussion of the Accident which Befell the Fakhrī Sextant (107:7 – 109:3)

Figure 13 in the text and translation is adequate for representing the effect of the drop in the member holding the aperture. It is faulty, however, inasmuch as the same circle is used to represent both the celestial sphere and the graduated arc of the sextant. In fact the radius of the first circle is assumed to be very large in comparison with that of the second so that all the rays GH, GK, and GM should have been shown as parallel instead of palpably diverging.

It is also implicit in Bīrūnī's remarks that the settling of the aperture must have occurred after the summer solstice observations and before the winter solstice, otherwise not only would H have been depressed, but also Z. The resultant ε would still have been smaller than it should have been, but the effect would have been much less pronounced than the one described.

29. Bīrūnī's Own Determinations of the Obliquity (109:4 – 111:14)

Having stated and assessed all previous obliquity observations known to him, the author turns to his own. First he reverts to work at Būshkānz already mentioned in 79:1–8. This village seems to have disappeared; it is mentioned nowhere else in the literature, and the best we can say is that it was south of Kāth (Section 8), to the west of the Oxus, and of latitude 41;36°.

In preparation for an extended campaign of observations, Bīrūnī had installed there a horizontal ring of diameter about 8.1 meters. But shortly after the summer solstice observation of 994 (the statement at 246:4 implies 995) civil disturbances attending a change in the Khwārazmian dynasty caused him to flee the country. He gives again the value of ε determined there, 23;35,45°.

He had intended also to obtain ε trigonometrically by using the solar altitude at the instant the sun was due east, the arc BL in Figure C6. Then from the proportions

$$TW/WY = YE/ZE \quad \text{and} \quad ZE/EP = TY/TW, \qquad (110:5)$$

ZE and EP (=Sin ε) could have been calculated (cf. Section 17). But he has lost the value of BL, obtained from the length of the east-west shadow (79:6), and can proceed no farther.

In 110:12 – 111:9 he goes through the same procedure with the Jurjāniya observation of 15 June 1016 already reported (79:9), this time exploited to find ε.

It is obtainable immediately from max $h = 71;18°$ (79:11) and $\bar{\varphi} = 47;42,10°$ (80:14).

$$\varepsilon = \max h - \bar{\varphi} = 23;35,50° .$$

Bīrūnī also calculates it trigonometrically. From the similar right triangles YWT and YZE on Figure C6 he uses (111:3)

$$WY / TW = ZE / YE$$

to obtain

$$ZE = \frac{WY \times YE}{TW} = \frac{\text{Cos } \widehat{KET} \times \text{Sin } \widehat{LEF}}{TK - LO} = \frac{19;14,12 \times 35;41,22}{21;8,35}$$

$$= \frac{8,897,635,464 \text{ fourths}}{76,115 \text{ seconds}} = 116,897 \text{ seconds}.$$

In fact he does not need ZE, and turns to the similar right triangles YWT and YPE in which

$$WY / TY = EP / YE.$$

So (111:7)

$$\text{Sin } \varepsilon = EP = \frac{WY \times YE}{TY} = \frac{8,897,635,464 \text{ fourths}}{102,904 \text{ seconds}} = 24;1,5.$$

the value of TY being obtained from 80:9.

The precise value of arc Sin 24;1,5 is 23;35,49°, but Bīrūnī writes 23;35,50°, the result obtained without trigonometry.

The work during 1016 was also terminated by political disturbances attendant upon the expansion of the Ghaznavid empire northward to include Khwārazm. The Abū al-ʿAbbās mentioned in 110:13 was a son and successor to the chief who had usurped the throne of the Khwārazmshāhs in 995 as mentioned above. The new dynasty was liquidated by Maḥmūd of Ghazna, under whose patronage, or at least his sufferance Bīrūnī was writing the Taḥdīd. It was possible for him to refer to Abū al-ʿAbbās as a "martyred prince" because the latter was not executed by Maḥmūd, but was murdered by his own soldiery for acceding to Maḥmūd's demands.

Last are his observations at Ghazna (111:10), meridian transits for the two solstices of 1019, and for the summer solstice of 1020. These give

$$\varepsilon = \frac{h_{max} - h_{min}}{2} = \frac{80;0^{\circ} - 32;50^{\circ}}{2} = 23;35^{\circ},$$

and

$$\varphi = 90^{\circ} - \frac{h_{max} + h_{min}}{2} = 33;35^{\circ}.$$

The same observations are described in the Canon, IV, 8 (vol. 1, pp. 404-8), translated in Schoy, Bestimmung.

Bīrūnī is not one to take seriously the ancient value, 24°, used by the Indians, but he feels obligated to discuss a statement by al-Makkī (see Section 23 above). The Indian astronomers did take parallax (112:7) into consideration in their computations, but this has nothing to do with their value of ε.

For the complicated story of the Sindhind zīj and its transmission to Baghdad, see Yaʿqūb. It was based ultimately upon the Brāhmasphuṭasiddhānta written by Brahmagupta in 628.

30. The Effect of Parallax (112:7 - 115:13)

Bīrūnī demolishes al-Makkī's allegation by showing that its actual effect is opposite to the direction of the difference between Indian and non-Indian values of ε. His argument may be restated by noting that 2ε is obtained by measuring (see Figure 14)

$$\text{angle } AHG - \text{angle } AHB,$$

if parallax is neglected, whereas it will be

$$\text{angle } AEG - \text{angle } AEB,$$

if parallax is taken into consideration.

Now, since

$$EM > EK > EZ,$$

hence $\quad$ angle $G >$ angle B, $\quad$ or $G - B > 0$,

and

$$AHG - AHB = (AEB + B) - (AEG + G) = (AEB - AEG) + (B - G) > AEB - AEG.$$

That is, if parallax is considered (as alleged for the Indians), the resulting ε will be less than the result obtained by neglecting parallax. In point of fact, however, the Indian value of 24° is larger than the Ptolemaic and Muslim determinations.

With the typical medieval penchant for enumerating special cases, Bīrūnī has separate discussions for the situations obtaining when $\varphi = \varepsilon$ (115:1), and when $\varphi = 0$ (115:7). Our Figures C15 and C16 (in contrast to their cognates in the text and the translation) have been so drawn as to exhibit the resemblance between the general configuration and the special cases. When $\varphi = \varepsilon$ the demonstration above is still valid. When $\varphi = 0$ the proof breaks down, but it is clear from Figure C16 that then also the angle at E is less than that at H. Bīrūnī says nothing about what happens when $\varepsilon > \varphi > 0$.

He drags in max β_m, the maximum lunar latitude (114:8), to strengthen a case already proved. Schematically,

	Indian Astronomy	Ptolemaic Astronomy
ε	24°	c. $23\frac{1}{2}^{\circ}$
max β_m	$4\frac{1}{2}^{\circ}$	5°.

Perhaps what he has in mind is that both parameters should be affected in the same way by parallax, but the Indian value of the one is larger, and of the other smaller than the Western values.

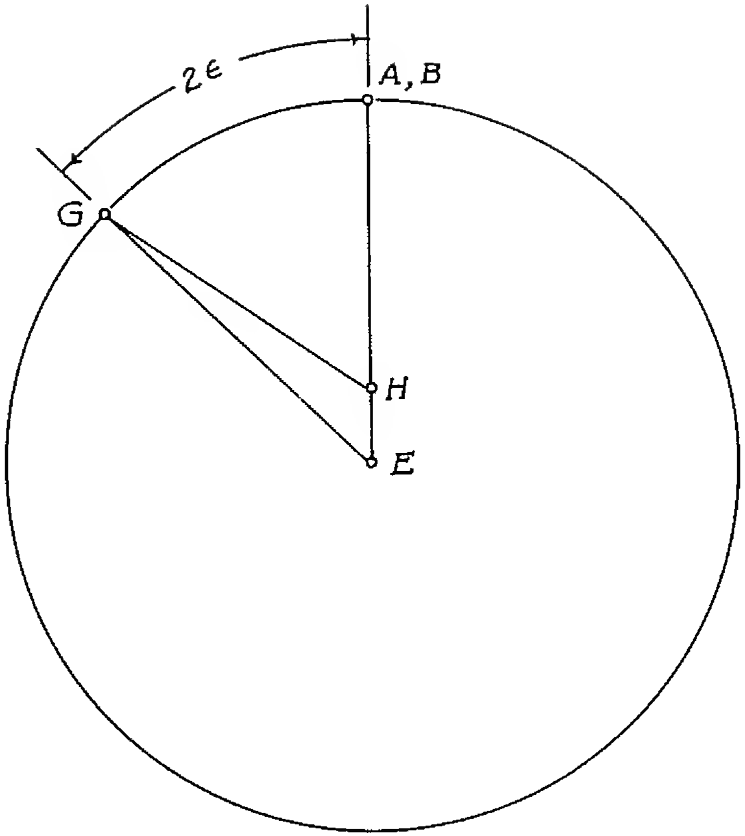


Figure C15

31. Conclusions (116:1 – 14)

Bīrūnī winds up the chapter by deciding upon 23;35° for the obliquity. It is the value he himself has found, and it is confirmed by many independent observers. Calculated by modern methods by Dr. David King, the value for his time (say 1020) was 23;33,59°, so his determination was just about a minute of arc too high. Evidently the original of the Taḥdīd contained a table listing all known determinations of ε, but there is no sign of it in the only copy extant. Such a list of Muslim values of the obliquity is to be found in Schirmer, pp. 60–61.

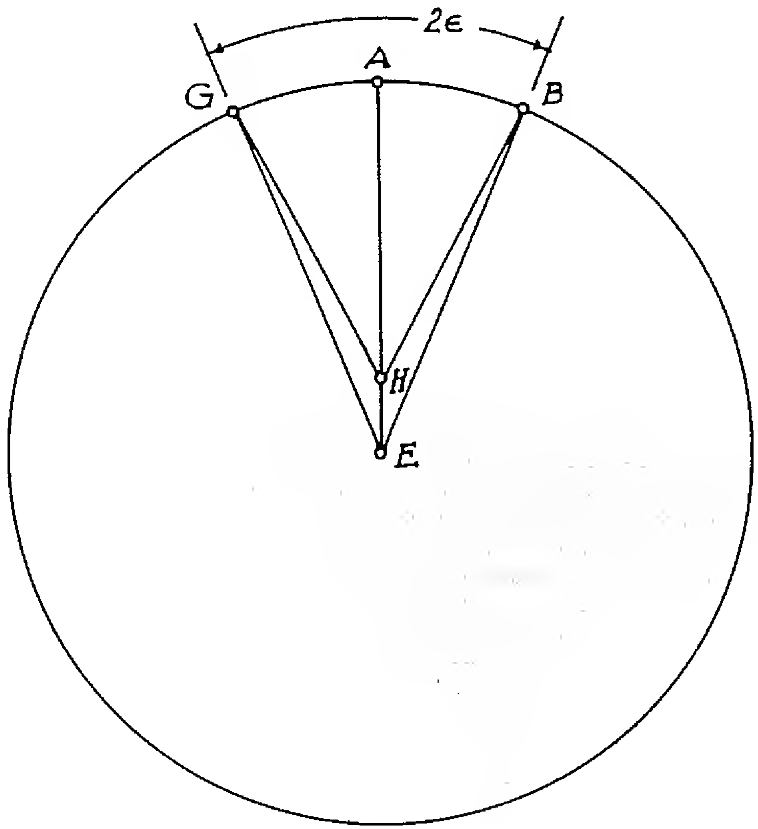


Figure C16

32. Relations Between Meridian Solar Altitude, Declination, and Local Latitude (117:1 - 121:9)

One of two simple arithmetical relations connects these three quantities. If the meridian altitude (h) is measured from the south, then

$$\varphi = \delta + h.$$

If we admit negative values of δ, which Bīrūnī could not, this expression will hold whether the sun is to the north of ZE ($\delta > 0$), as at T in Figure 17 (118:4), or when it is to the south ($\delta < 0$), as at H.

If h is measured from the north, then

$$\varphi = \delta - h,$$

the case illustrated by point K.

Clearly if φ is known, as well as either δ or h, the third quantity falls immediately out of either of the two expressions above which happens to apply.

Three worked examples are given. The first (119:1) is based on an observation Bīrūnī made on 14 October, 1018, in the vicinity of Kābul, capital of modern Afghanistan. The work was done under very trying circumstances and is illustrative of Abū Rayḥān's devotion to science. This was shortly after the extinction of the Khwārazmshāh dynasty, the violent death of Bīrūnī's king and patron, and the deportation of our author, together with other local notables. Bīrūnī was doubtless being taken south to the grim Maḥmūd's capital at Ghazna. On the way, he seized the opportunity to determine the latitude of Kābul by measuring the sun's meridian transit (h) with an instrument improvised on the spot from a (dust abacus?) calculating board. With the aid of Battānī's zīj he calculated δ the solar declination at the time, obtaining $\delta(\lambda_s) = \delta(180^\circ + 26;36^\circ) = 10;19^\circ$. This zīj is extant, and the result was verified by use of its declination table (Battānī, vol. 2, p. 57). Hence

$$\varphi = 90^\circ - \bar{\varphi} = 90^\circ - (h - \delta)$$
$$= 90^\circ - (45^\circ - (-10;19^\circ)) = 34;41^\circ.$$

The modern value for the latitude of Kābul is 34;30.

The second example was ordered by the wazīr Ibn al-ʿAmīd referred to in Section 9 above, and the work was recorded in al-Khāzīn's Ṣafā'iḥ Zīj (Section 12 and 25). The observation took place

in Kāshān, a city of central Iran, on 6 October, 960. According to the zīj the solar declination was then

$$\delta(\lambda_s) = \delta(180^\circ + 18;37^\circ) = -7;20^\circ.$$

Use of Bīrūnī's own declination table (Canon, vol. 1, p. 374) gives the same result. Thence

$$\varphi = 90^\circ - \bar{\varphi} = 90^\circ - (h - \delta) = 90^\circ - (50^\circ + 7;20^\circ) = 32;40^\circ.$$

The modern value for the latitude of Kāshān is 33;59°, so that Bīrūnī's doubts about the accuracy of this determination are well founded.

The third example is a meridian solar altitude observed by Bīrūnī at Jurjāniya on 17 September, 1016. The text has h = 47;42°, but from the rest of the passage it is clear that 47;44° was intended.

$$\delta = h - \bar{\varphi} = 47;44^\circ - 47;43^\circ = 0;1^\circ.$$

The book to which he refers in 121:2 is not extant. In Boilot (RG 101) the title appears as Kitāb al-taṭbīq, but in the Chords (Hyd. ed., p. 69) it is given as taṭrīq, as in the Taḥdīd, and this is probably correct.

33. The Relation Between Declination, an Altitude of Azimuth Zero, and the Local Latitude (121:10 - 122:12)

This special situation resembles the one just discussed only to the extent that the altitude is observed as the sun passes through a cardinal direction -- the essential simplicity of the plane configuration which suffices for a meridian transit is lost, and spherical trigonometry must be called upon.

The solution is quickly inferred from Figure C18, in which LO is a vertical line, OF and LF are perpendicular to ZD, and OK is perpendicular to LF. The desired relation is (121:17)

$$\frac{\text{LO } (=\text{Sin } h)}{\text{OK } (=\text{Sin } \delta)} = \frac{\text{Sin LKO } (=R)}{\text{Sin } (\text{OLK} = \varphi)},$$

or

$$\sin \varphi = \sin \delta / \sin h.$$

Given any two elements of the triple, φ, δ, and h, the third may be calculated.

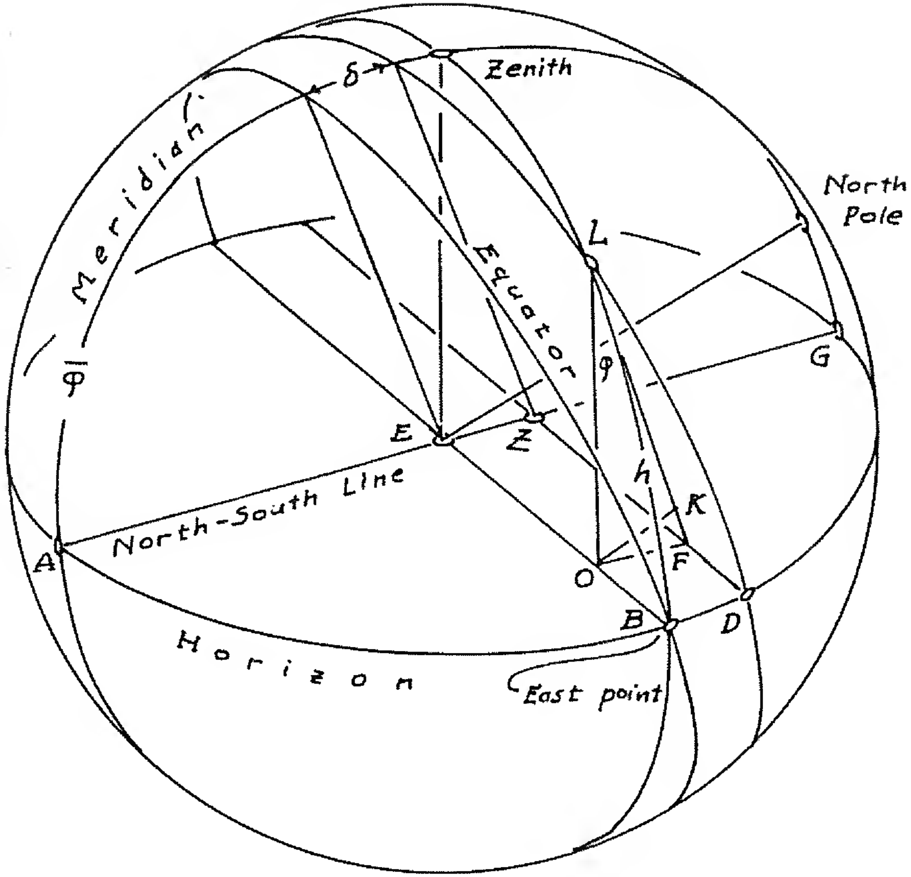


Figure C18

34. Relations Between Local Latitude, Declination, Altitude, and Azimuth (122:13 – 127:8)

The problem discussed just above is a special case of this one. Assume first that it is desired to find φ and the other three quantities (δ , h, and az.) are known. Referring to any of the parts of Figure C19, we have, by similar triangles (123:2)

$$\frac{EO\ (= \text{Cos } h)}{OC\ (= \text{share of az.})} = \frac{EM\ (=R)}{\text{Sin }(BM = \text{az.})}$$

So the share of the azimuth is

$$OC = (\text{Cos } h)\ (\text{Sin az.}) \,/\, R$$

Note that the plane of O, F, and L is parallel to the meridian plane. Moreover, OF is in the horizon and LF is in the plane of the day-circle. Hence angle LFO = $\bar{\varphi}$, and, LO being vertical, angle OLF is the desired φ . Draw CK perpendicular to LF.

By the Pythagorean proposition (123:7),

$$LC = (\overline{LO}^2 + \overline{OC}^2)^{\frac{1}{2}} = (\text{Sin}^2 h + \overline{OC}^2)^{\frac{1}{2}} ,$$

in which OC has just been found.

In the right triangle LOC (123:8)

$$\frac{LC}{OC} = \frac{\text{Sin }(LOC = 90^{\circ})}{\text{Sin } OLC} .$$

Hence angle OLC, "the first arc", can be calculated.

Also, in right triangle CLK (123:13)

$$\frac{CK\ (= \text{Sin } \delta)}{CL} = \frac{\text{Sin } CLK}{\text{Sin }(CKL = 90^{\circ})} .$$

Now CL has just been expressed in terms of knowns, hence "the second arc", angle CLK, can be calculated.

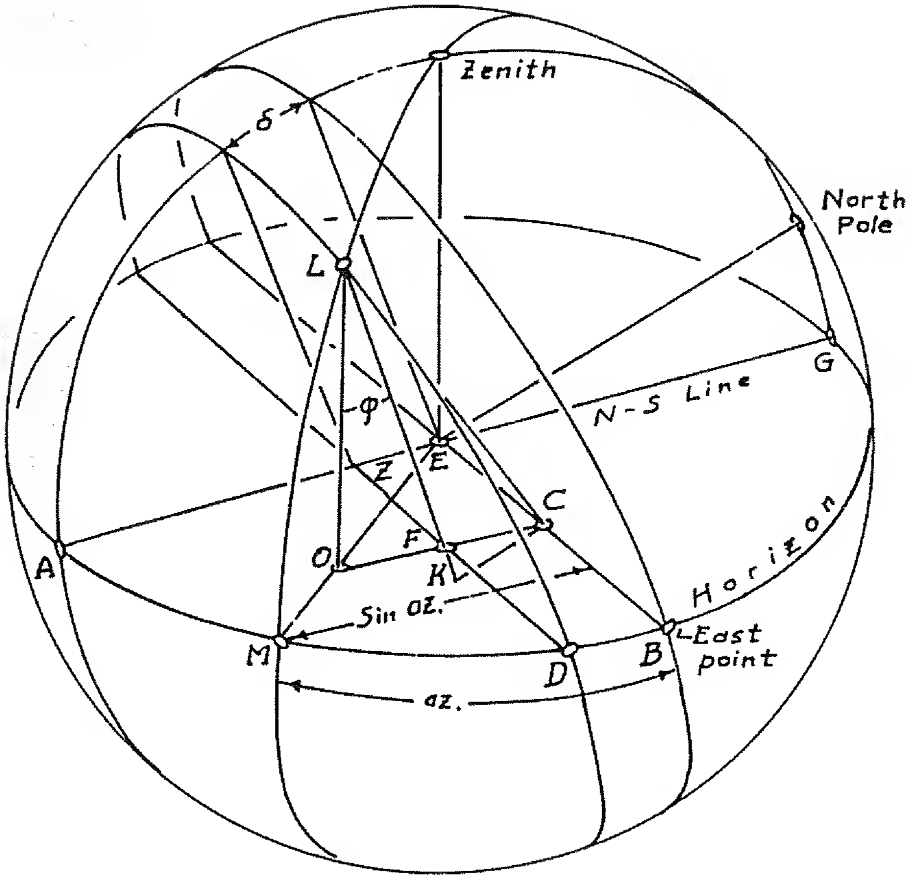


Figure C19.1

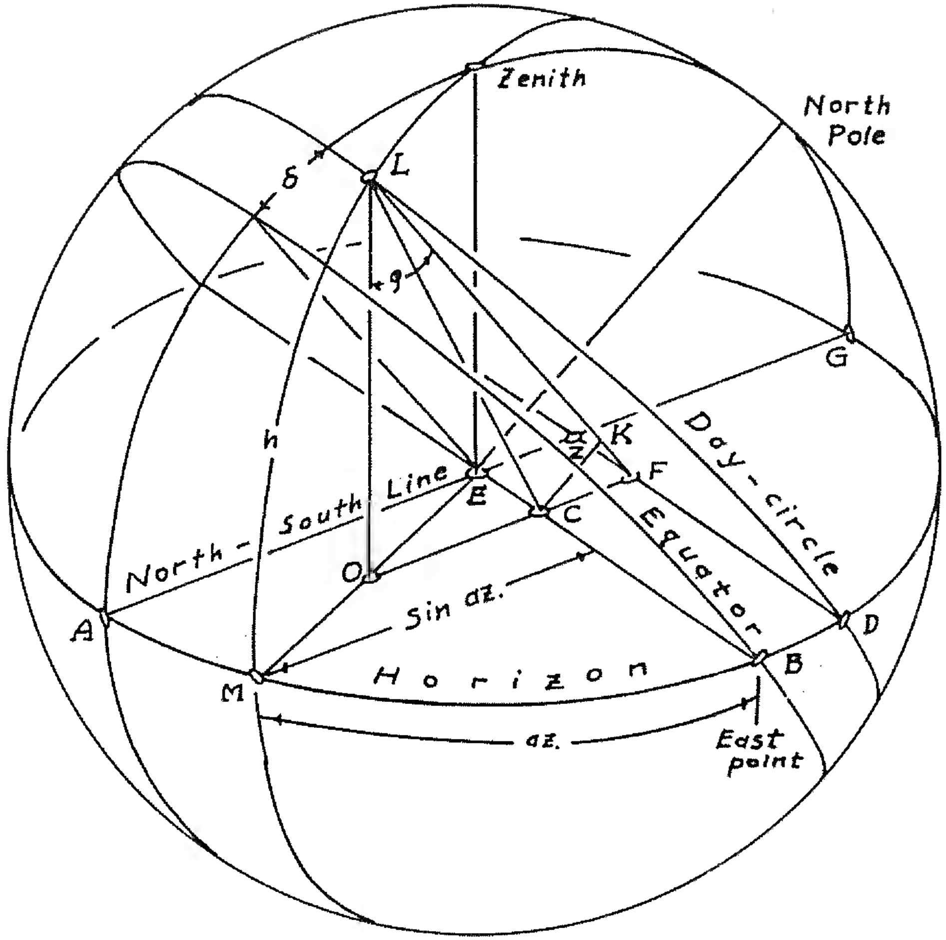


Figure C19.2

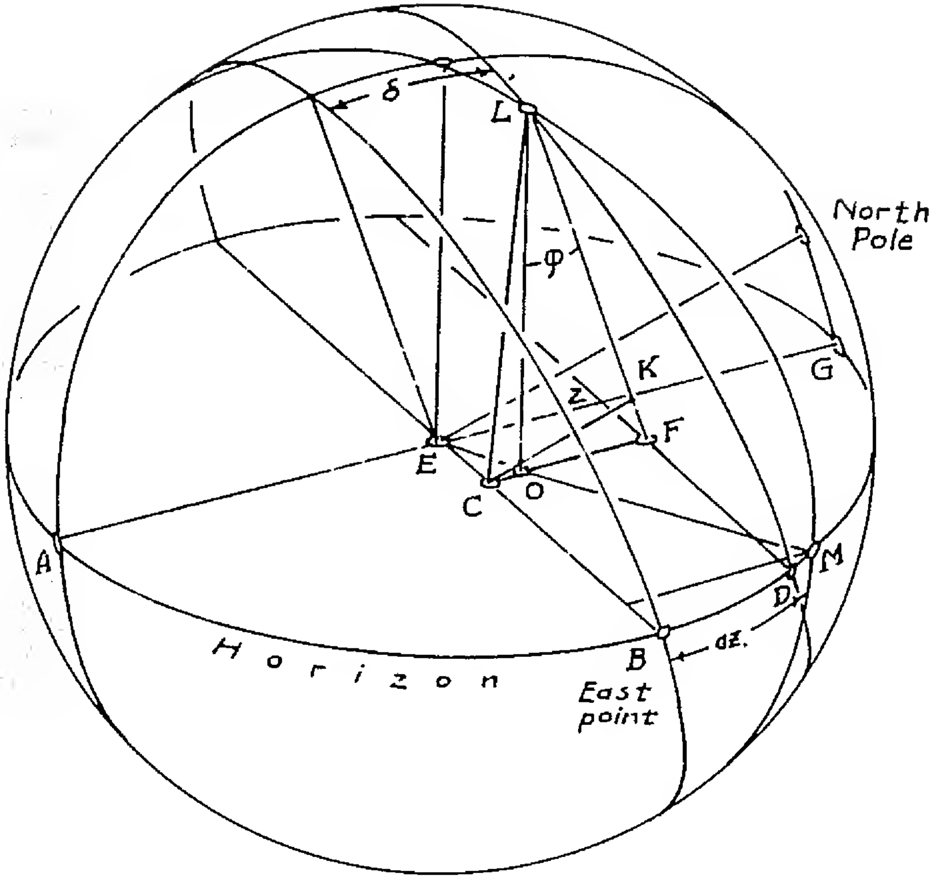


Figure C19.3

If $\delta_i < 0$ and the azimuth is south (123:17, Figure C19.1),
φ = angle OLC − angle CLF = 1st arc − 2nd arc.

If $\delta_i > 0$, and the azimuth is south (123;19, Figure C19.2)
φ = angle OLC + angle CLF = 1st arc + 2nd arc.

If $\delta_i > 0$ and the azimuth is north (123:21, Figure C19.3)
φ = angle CLF − angle OLC = 2nd arc − 1st arc.

Bīrūnī seems to have been misled by his figure. The angle OLK is not obtuse in any of the cases considered.

Whenever $\delta = 0$ (125:3), point F merges with C, the second arc becomes zero and φ is the first arc.

If φ and the horizon coordinates of the sun are assumed known (125:5), δ can be calculated. The share of the azimuth, OC, is determined as in 123:8. Then, since (125:7)

$$\frac{\text{LO } (=\text{Sin } h)}{\text{OF}} = \frac{\text{Sin } (\text{OFL} = \bar{\varphi})}{\text{Sin } (\text{OLF} = \varphi)} \quad (=\cot\varphi),$$

$$\text{OF} = \text{Sin } h \cdot \text{Sin } \varphi \,/\, \text{Cos } \varphi \,.$$

Here again Bīrūnī fails to use the tangent function, when he could have saved some time and trouble by employing it (cf. Section 17).

Now

$$\text{CF} = \begin{cases} \text{OC} - \text{OF}, & \text{Figure C19.1} \\ \text{OF} - \text{OC}, & \text{Figure C19.2} \\ \text{OC} + \text{OF}, & \text{Figure C19.3} \end{cases}$$

In right triangle CFK (125:12)

$$\frac{\text{CF}}{\text{CK } (=\text{Sin } \delta)} = \frac{\text{Sin } (\text{CKF} = 90^{\circ})}{\text{Sin } (\text{KFC} = \bar{\varphi})},$$

so $$\text{Sin } \delta = (\text{CF} \cdot \text{Cos } \varphi) / R.$$

Next (125:15) are directions for calculating the azimuth in terms of φ_i, δ_i, and h. First, use the proportion just above to calculate

$$\text{CF} = (\text{Sin } \delta)\, R \,/\, \text{Cos } \bar{\varphi}_i \,.$$

Thence the Pythagorean Theorem gives

$$\text{KF} = (\overline{\text{CF}}^2 - (\text{CK} = \text{Sin } \delta)^2)^{\frac{1}{2}} \,.$$

Here again the use of the tangent function in the expression,

$$KF = \text{Sin}\ \delta\ .\ \tan\varphi$$

would have been simpler.

In the similar triangles KFC and FOL (126:2),

$$\frac{KF}{KC\ (=\text{Sin}\ \delta)} = \frac{FO}{OL\ (=\text{Sin a})},$$

so $$OF = KF \cdot \text{Sin h} / \text{Sin}\ \delta\ .$$

Now (126:4) the share of the azimuth is

$$OC = OF \pm CF,$$

the plus sign being used when $\delta < 0$. By similar triangles,

$$\frac{OC}{\text{Cos}\ \ h\ (=OE)} = \frac{\text{Sin az.}}{R\ (=ME)},$$

whence $$\text{Sin az.} = R \cdot OC / \text{Cos h}.$$

Up to this point, Bīrūnī has used plane trigonometric methods upon figures constructed in the interior of the sphere. For the fourth and last situations, however (δ , φ , and azimuth given, to calculate h), he operates entirely on the surface of the sphere, and with great circle arcs. Referring to Figure C20.1, and applying the Rule of Four to the right spherical triangles BOC and BGZ which have angle B in common (126:15),

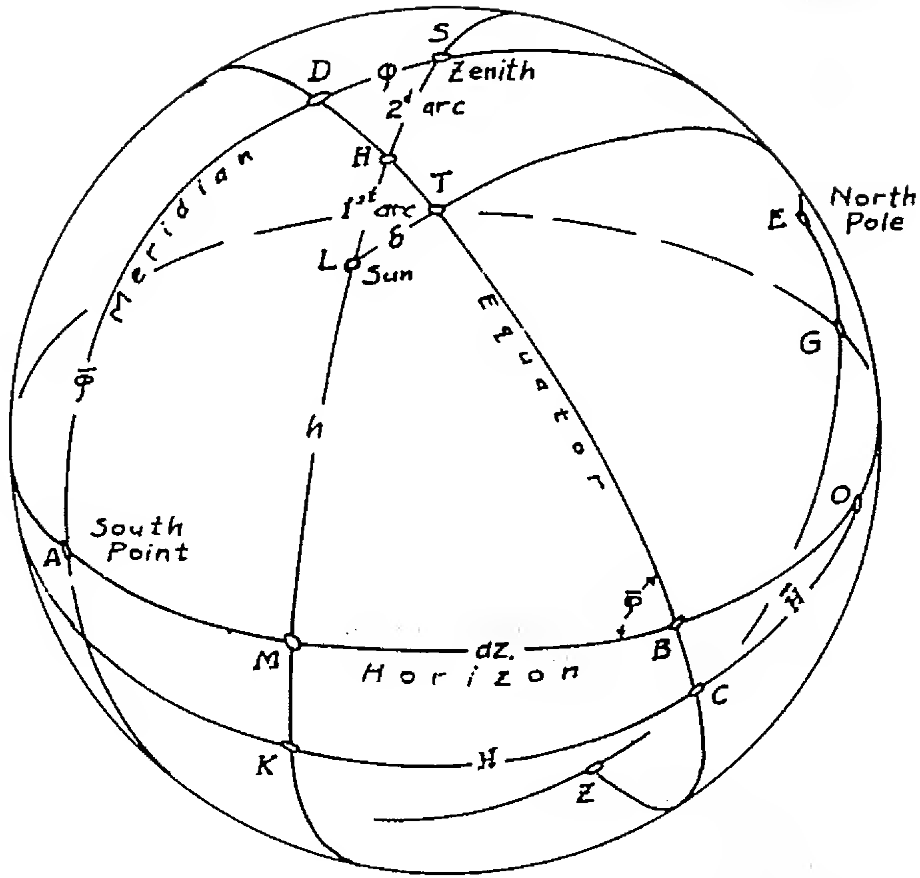


Figure C20.1

$$\frac{\text{Sin (BO} = \overline{\text{az}}.)}{\text{Sin (OC} = \overline{\text{KHC}})} = \frac{\text{Sin BG} (=\text{R})}{\text{Sin (GZ} = \bar{\varphi})} .$$

Hence

$$\text{Cos KHC} = (\text{Cos az.} \cdot \text{Cos } \varphi) \text{ R}.$$

Also, by the same rule applied to triangles LHT and KHC (126:19),

$$\frac{\text{Sin (HL} = \text{1st arc)}}{\text{Sin (LT} = \delta)} = \frac{\text{Sin (LTH} = 90^{\circ})}{\text{Sin KHC}} ,$$

whence $\text{Sin (1st arc)} = (\text{Sin } \delta) \text{ R / Sin KHC}.$

Further (127:1),

$$\frac{\text{Sin (HS} = \text{2nd arc)}}{\text{Sin (SD} = \varphi)} = \frac{\text{Sin (SDK} = 90^{\circ})}{\text{Sin (KC} = \text{KHC)}} ,$$

by the Rule of Four applied to triangles HSD and KHC. So

$$\text{Sin (2nd arc)} = (\text{Sin } \varphi) \text{ R / Sin KHC}.$$

In the case shown in Figure C20.1, the required h is found from the relation (127:5)

$$\bar{h} = \text{1st arc} + \text{2nd arc},$$

but when $\delta > 0$ (as in the upper right and lower left of Figure 20), $\bar{h}$ is the difference between these two arcs.

35. Solar Declination from Two Azimuths and Altitudes (121:1–14)

In a previous passage (72:13–75:7) the author has demonstrated the computation of φ in terms of two observed sets of solar horizon coordinates. With the same data at hand, he now explains a method for calculating δ. References are to Figure C21, although the configuration and lettering are essentially those of Figure C4.

In the similar right triangles TWY and TKN (128:6),

$$\text{TW / WY} = \text{TK / KN}.$$

In this proportion TW = Sin h_2 − Sin h_1, and the determination of WY has been explained in 73:13 (see Section 17). TK = Sin h_2, so KN is the only unknown and can be calculated. The

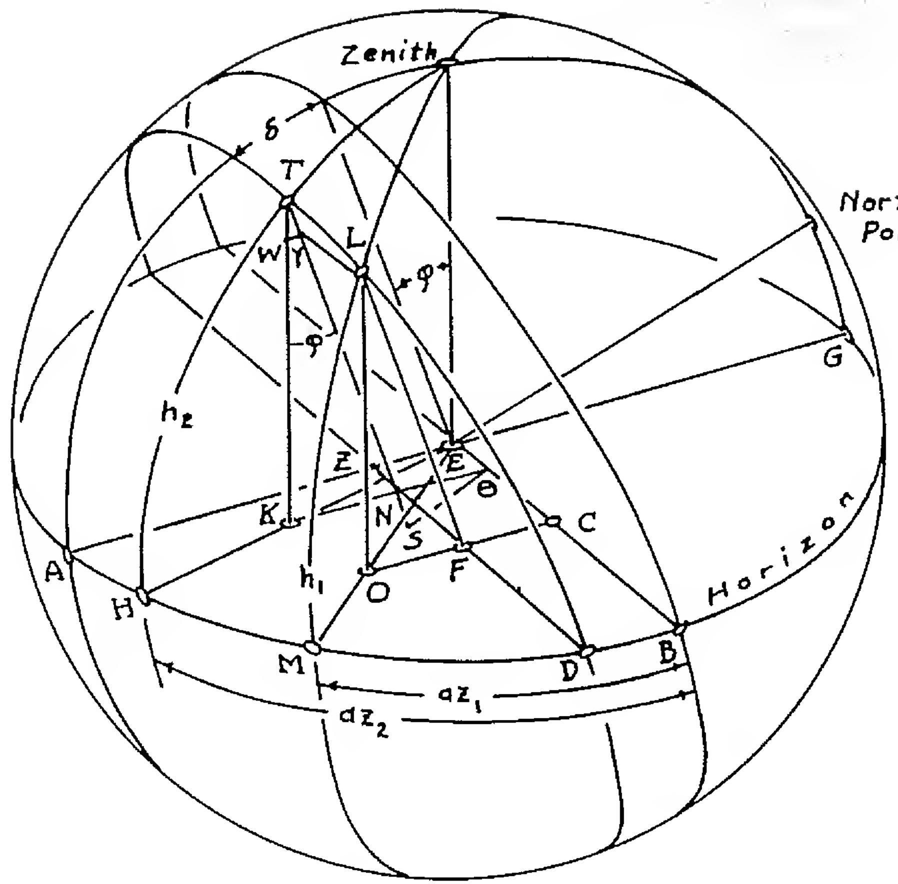


Figure C21

determination of $K\theta$ has also been explained in 73:13, so θN = $K\theta - KN$ = Sin DB is obtainable. (Arc DB is the rising amplitude.) θS has been drawn perpendicular to TN (128:9). It is therefore the distance between the parallel planes of the equator and the day-circle, hence the sine of the desired declination. By similar triangles (128:12),

$$\frac{N\theta\ (= \text{Sin DB})}{\theta S\ (= \text{Sin } \delta)} = \frac{TY}{TW}$$

The elements of the second ratio have been determined as explained in 73:15. Hence (128:13)

$$\text{Sin } \delta = (\text{Sin DB}) \cdot TW / TY$$

can be calculated, which solves the problem.

36. A Worked Example of the Above (129:1 - 16)

Bīrūnī's own observations of 7 December 1016 at Jurjāniya, previously used (75:8-76:9) to calculate the latitude of that place are now employed to illustrate the calculation of δ.

Using the calculations of 75:17 and 76:2, the difference between the "shares of the azimuths" is (129:4)

$$WY = K\theta - DC = 51;41,35 - 46;0,53 = 5;40,42.$$

Also, from 76:4, the sine of the larger altitude is Sin h_3 = 21;39,54, and the difference between them is TW = Sin h_2 - Sin h_1 = 6;18,16. Hence, (cf. 128:6),

$$\begin{aligned} KN &= 5;40,42 \times 21;39,54 / 6;18,16 \\ &= 1,594,353,348 \text{ fourths} / 22,696 \text{ seconds} \\ &= 19;30,48 \end{aligned}$$

Now N = K - KN = 32;10,47 = Sin DB (129:7), and TY, from 76;7, 30,545 seconds, so (from 128:13),

$$\begin{aligned} \text{Sin} &= (\text{Sin DB})\ TW/TY \\ &= 2;629,263,512 \text{ fourths}/30,545 \text{ seconds} \\ &= 86,078 \text{ seconds} \\ &= 23;54,38. \end{aligned}$$

This corresponds (129:9) to

$$\delta = 23;29,6^\circ .$$

37. Latitude of Jurjāniya from Symmetrically Disposed Meridian Altitudes (129:10 - 130:12)

Instead of using a pair of solsticial meridian altitude observations, it is possible to compute $\bar{\varphi}$ by taking the arithmetic mean of a pair of altitudes taken when the sun is in each of a pair of points equidistant from an equinoctial point.

Bīrūnī gives two such determinations for Jurjāniya, but for the first he simply reports the result, leading to a φ of 42;17° (129:15

For the second, full preliminary data are reported. On Sunday 2 September 1016, the meridian altitude was h = 53;35°. From the zīj of Ḥabash, Bīrūnī calculates that λ_s was then Virgo 15;11° He seeks the altitude the sun would have had if it had crossed the meridian when it was at the midpoint of Virgo. The meridian altitu on the preceding day was 53;58°. From this he concludes that the desired altitude was 53;36° (130:6).

We are unable to verify this determination. The change in altitude (see Figure C21.1) equals the change in declination, and this is as it should be, for the daily $\Delta\lambda$ is of the order of one degree, and in that part of the ecliptic $\delta(\lambda + 1°) - \delta(\lambda)$ is indeed 0;23° to the nearest minute of arc (from the <u>Canon</u>, vol. I, p. 373). The δ function is nearly linear in this region, so linear interpolation is reliable. The correction x to be added to the Sunday altitude is therefore

$$\frac{x}{0;23} = \frac{0;11}{1;0}$$

whence x = 0;4, and the corrected altitude should have been 53;39°

Bīrūnī also made observations from which he could calculate the meridian altitude for the midpoint of Libra, the sign which, with Virgo, straddles the autumnal equinox. For Tuesday, 2 October 101 the meridian altitude was 41;53, and the following noon it was 41;30 (130:7). By computations based on the same zīj, λ_s is Libra 14;51°. This time the correction, x = 0;23 · 0;9 = 0;3, is to be subtracted, and the corrected altitude is 41;50°, in contrast to the text's 41;52° (130:9). Our results give $\bar{\varphi} = \frac{1}{2}(53;39° + 41;50°)$ = 47;44;30°, which is close to Bīrūnī's 47;44° (130:11, restored). The purely fortuitous near symmetry of the two pairs of noons with respect to the autumnal point is the reason for this closeness.

The zīj Bīrūnī uses here was the work of Ḥabash al-Ḥāsib al-Marwazī (fl. 840, see <u>Suter</u>, p. 12, and the <u>Survey</u>), a prominent astronomer of Baghdad. To him different zījes are attributed, and two versions of such tables are extant.

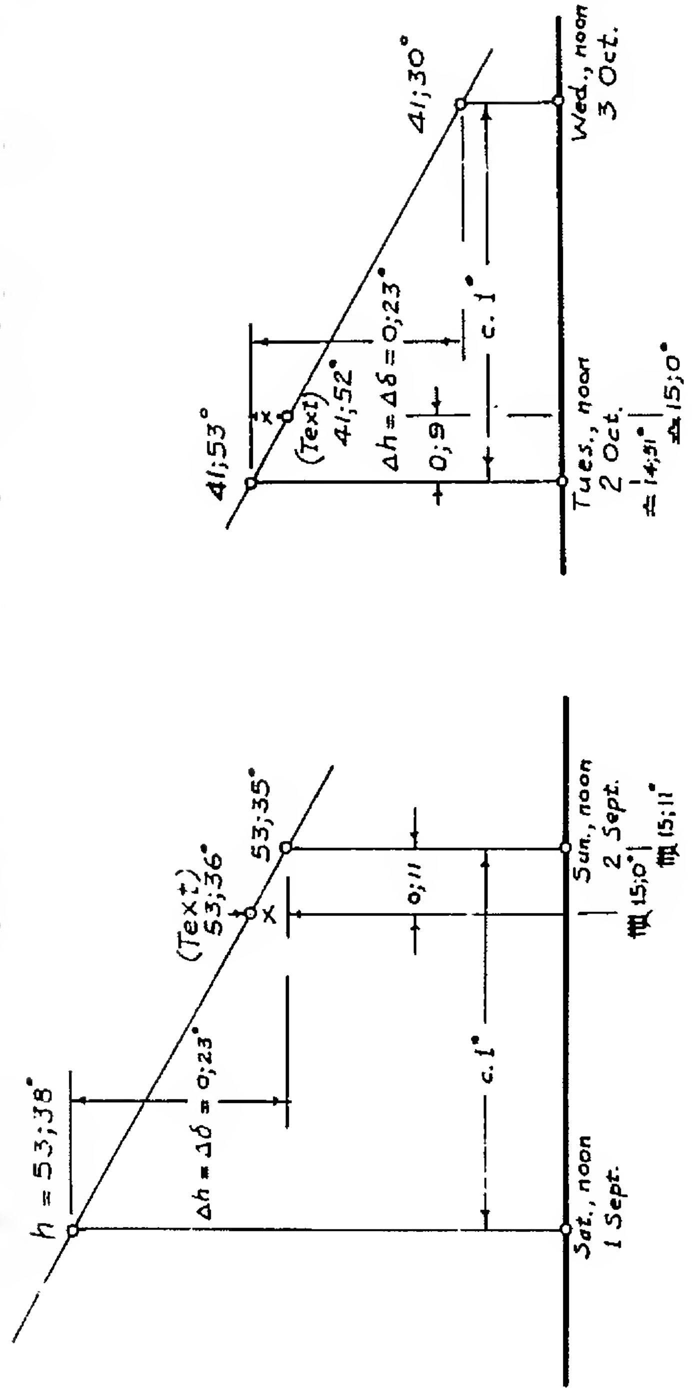


Figure C21.1

38. An Instrument for Local Latitude (130:13 – 131:14)

It is assumed that a meridian line on a horizontal plane is available, and that the sun's declination on the day of the observation is known. A gnomon of length R is erected at the center of a rectangular board. The units suggested for R: 12 digits, 6 1/2 feet, and 60 parts (130:15), were all standard in medieval astronomy, and tables of the corresponding functions were available (cf. e.g. <u>Shadows</u>, f. 205 v of the unique Patna MS. Missing in the edition). The objective will be attained if the board can be fixed in the equatorial plane, for then the angle between it and the horizontal will give the complement of the latitude. A first step is to fix one edge of the board, by hinges, say, along the east-west line, for the intersection between the horizontal and equatorial planes is such a line. To fix an additional line on the board will suffice to fix its position. To do this, draw on it a circle of radius $\mathrm{Cot}_R\,\delta$ as shown on Figure C21.2. Then rotate the board until the shadow of the gnomon's tip falls upon the circle. This ensures that the angle between the sun's rays and the plane of the board shall be δ, which is a second requirement of the equatorial plane. Of course, when $\delta < 0$, gnomon and circle must be on the underside of the board.

When δ is small, i.e. in the vicinity of an equinox, the method is impractical, since then the radius of the circle is large. Furthermore, there is no point in drawing a whole circle; a semicircle with center on the upper edge of the board would serve as well or better. Something of this sort may have been in the author's mind, for in the succeeding passage describing what to do when the meridian is unknown, he says (131:11) <u>half the board</u>.

This procedure (131:11 – 14) to be followed in the absence of a predetermined meridian is in any event incomplete as given — one observation during the day is insufficient. An addition to the text so that the translation read

> ". . . we seek a position for setting up the board such that <u>at all times during the day</u> the shadow falls on the circumference of the circle . . ."

would remedy matters. However, to make the rest of the procedure valid it is necessary to add that a pair of opposite sides of the board must be horizontal, otherwise the feet of the two plumb lines will not define the meridian.

Like the devices commented upon in Section 16 above, this instrument cannot have been intended for serious observations. At best, it might be useful for didactic purposes.

39. Relations Between Latitude (φ), Solar Declination (δ), Rising Amplitude (r), and Daylight Length (131:15 – 133:19)

Referring to Figure C22, we note that the length of daylight in degrees of daily rotation is $2\,(90^\circ + q)$, where q is the equatorial arc EK, the equation of daylight. Hence all four of the quantities dealt with in this section: φ , δ , r, and q , are acute angles or sides of the right spherical triangle EKH. It follows that if any two of them are known, the remaining two can be found.

Applying to the triangles EFH and EDG the Rule of Four (132:4)

$$\frac{\text{Sin}\,(HE = r)}{\text{Sin}\,(HK = \delta)} = \frac{\text{Sin}\,(ED = 90^\circ)}{\text{Sin}\,(GD = \bar{\varphi})}$$

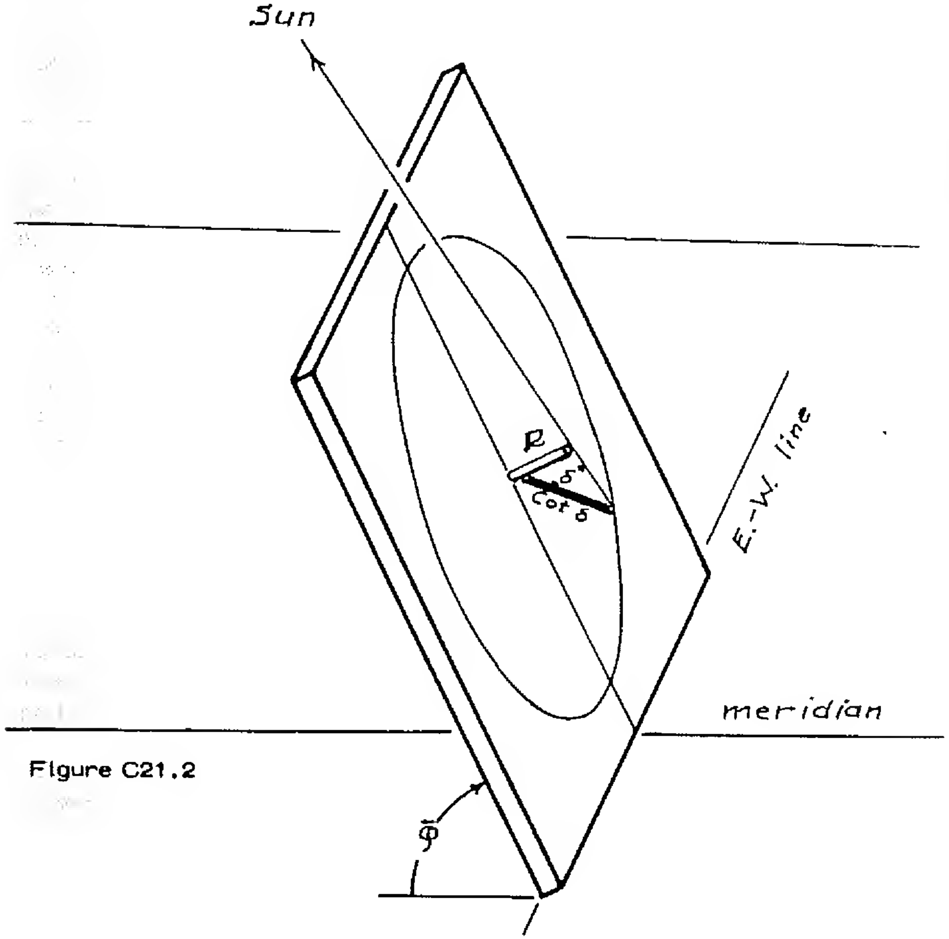


Figure C21.2

Hence,

(1) $(\text{Sin}\,r \cdot \text{Cos}\,\varphi)/R = \text{Sin}\;\delta(\varphi, r)$,

where the parentheses on the right indicate that δ is found in terms of φ and r, and

(2) $\text{Sin}\,\delta \cdot R \,/\, \text{Sin}\,r = \text{Cos}\,\varphi(\delta, r)$.

The great circle LMSC has H as pole. Now M and K are right angles, hence C is the pole of MTK. So CK is a quadrant, and since E is the pole of ATD, EA is also a quadrant. Therefore, (132:15) AC = EK. Moreover, since D and L are both right angles, S is the pole of the horizon, the zenith. So (132:15) SA = DT = φ .

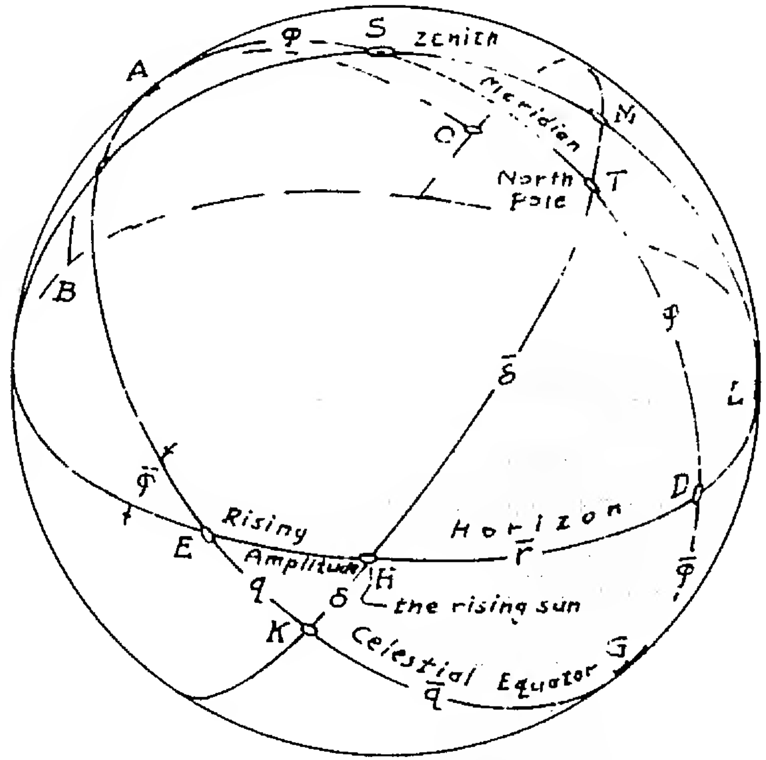


Figure C22

By the same theorem applied to triangles TSM and TAO,

$$\frac{\text{Sin }(TS = \bar{\varphi})}{\text{Sin } SM} = \frac{\text{Sin }(TA = 90^o)}{\text{Sin }(AO = GK = \bar{q})} .$$

(AO and GK are equal because they are the measures of vertical angles at T.) Hence

$$SM = \text{arc Sin}\left[(\text{Cos } \varphi \, . \, \text{Cos } q)/R\right],$$

and SC = $\overline{SM}$ can be found (132:18). But by the same theorem, in the triangles SCA and STM (132:18)

$$\frac{\text{Sin } SC}{\text{Sin }(AC = EK = q)} = \frac{\text{Sin }(ST = DG = \bar{\varphi})}{\text{Sin }(MT = HK = \delta)} .$$

Hence

(3) $\text{Sin } q \, . \, \text{Cos } \varphi / \text{Cos } SM = \text{Sin } \delta(\varphi, q)$.

In triangles THD and TKG, by the same theorem (133:4)

$$\frac{\text{Sin }(TH = \bar{\delta})}{\text{Sin }(DH = \bar{r})} = \frac{\text{Sin }(TK = 90^o)}{\text{Sin }(KG = \bar{q})} .$$

So

(4) $\text{Cos } \delta \, . \, \text{Cos } q / R = \text{Cos } r(\delta, q)$.

Applying the usual theorem to triangles HEK and THD (133:6)

$$\frac{\text{Sin }(HE = r)}{\text{Sin }(EK = q)} = \frac{\text{Sin }(TH = \bar{\delta})}{\text{Sin }(TD = \varphi)} ,$$

or

(5) $\text{Cos } \delta \cdot \text{Sin } q / \text{Sin } r(\delta, q) = \text{Sin } \varphi(\delta, q)$

In triangles THD and TKD (133:12)

$$\frac{\text{Sin }(TH = \bar{\delta})}{\text{Sin }(HD = \bar{r})} = \frac{\text{Sin }(TK = 90^o)}{\text{Sin }(KG = \bar{q})} ,$$

whence

(6) $\cos r \cdot R / \text{Cos } q = \text{Cos } \delta(r, q)$

(7) $\text{Cos } \delta(r, q) \cdot \text{Sin } q / \text{Sin } r = \text{Sin } \varphi(r, q)$

This completes the passage, and we can recapitulate its results by indicating, for each of the seven numbered equations above, which of the four quantities is expressed in terms of which of the remaining ones. They are

(1) $\delta(\varphi, r)$

(2) $\varphi(\delta, r)$

(3) $\delta(\varphi, q)$

(4) $r(\delta, q)$

(5) $\varphi(\delta, q)$

(6) $\delta(r, q)$

(7) $\varphi(r, q)$

This does not exhaust the possibilities, there being all told three such expressions for each of the four quantities.

We note that the Rule of Four for sines is the only theorem invoked throughout, as is the sine the only trigonometric function. Only arcs of great circles are involved, never spherical angles as such.

40. The Seven Iranian Keshvars (134:1 – 135:15)

The concept of the inhabited world as being composed of seven great regions or nations is set forth in a number of sources. In the Tafhīm (ed. of Wright, p. 142) Bīrūnī gives the same rosette diagram which appears on p. 136 of the Taḥdīd, but with only one name shown in each circle. These are displayed in the first column of the table below, except that the second keshvar (or kišvar) is named al-Maghrib (the West, or North Africa). The Tafhīm attributes the arrangement to Hermes (see Section 69) via "the Persians".

Al-Mas'ūdī (Murūj, p. 181) writing in 944 A.D., gives essentially the same arrangement, with a slightly different numbering, associating with each region (which he calls an iqlīm, climate), a planet.

Ḥamza al-Isfahānī (Reinaud, p. cxxiii, see Section 45) does not number the regions, but he associates directions with them. Following this lead, were they to be arranged in a rosette, a rotation of the result by sixty degrees counterclockwise would carry Ḥamza's arrangement into the standard one.

The famous astrologer Abū Maʿshar (Albumasar, fl. 850) and his pupil al-Kindī give pretty much the same list of nations, but in a somewhat different order. The planets which, they say, the Persians associate with the respective peoples are as shown in the column under al-Masʿūdī, except that the sun and the moon have switched places (Honigmann, p. 142).

The keshvars are undoubtedly of Iranian origin, dating from Sasanian times, possibly before, and connected with the seven dvīpas of the Indian Purānic cosmology.

Taḥdīd and Tafhīm			Masʿūdī			Ḥamza		Abū Maʿshar and al-Kindī	
Direction	No.	Name	No.	Name	Planet	Direction		No.	
SE	1	India	2	India	Saturn	SE	Chinese	1	India
S	2	Arabia	3	Arabia	Venus	S	Indians	5	Arabs
SW	3	Egypt, Syria	4	Egypt Africa	Mercury	SW	Negroes	6	Egypt
Center	4	Irānshahr	1	Babylon etc.	Jupiter	Center	Iranians	2	ʿIrāq
NW	5	Rome	5	Rome, Syria	Moon	NW	Berbers	4	Rome
N	6	The Turks	6	The Turks	Mars	N	Rome	3	Turks
NE	7	China	7	China	Sun	NE	Turks	7	China

41. Localities in the First and Fourth Keshvars (136)

In the first keshvar:

Sind is the region of the lower Indus valley.

Daybul, shown as connecting the first and fourth keshvars, was the main port of Sind, at the principal mouth of the Indus (LeStr., p. 331).

Zābij, see Section 2.

Zanj connotes the whole eastern coast of Africa, whence Zanzibar, an island just off the coast (Ḥudūd, p. 472).

In the fourth keshvar:

Al-Jabal, more commonly Jibāl, is the mountainous central highland of Iran. In Persian it is Kūhistān, whence the appellation of al-Kūhī (Section 24) a native of the region.

Khurāsān at present is the northeastern province of Iran, but in medieval times the name was applied as well to all the region south of the Oxus and west of Badakhshān. Ṭukharistān was that portion of Khurāsān just south of the Oxus from Balkh to Badakhshān (LeStr., 426-7).

Sijistān, the land of the Šaka, or Scythes, is modern Sīstān, the region just south of Khurāsān. In Sijistān the highlands northwest of the Helmand River were called Zabulistān (LeStr., p. 334).

42. Localities in the Fifth, Sixth, and Seventh Keshvars (136)

In the fifth keshvar:

Franja, the land of the Franks, is western Europe.

Burjān was a name applied to the Bulgars of the Danube (Ḥudud, p. 423).

Āzarbaijān comprised the territory of the northwestern province of modern Iran plus the adjoining Soviet republic having the same name.

In the sixth keshvar:

Gog and Magog are mythical peoples mentioned in the Bible and the Quran who are supposed to inhabit the unknown northeastern regions of the ancient world (EI, vol. iv, p. 1142).

For the Khazar and the Ghuzz Turks, see Section 8.

The Khirkhiz, or Kirgiz, or Qirghiz, is an ancient Turkic people which set up a nomad empire in the ninth century A.D. in the upper reaches of the Yenisei River. The latter flows north through central Siberia. Present habitat of the Kirgiz is midway between Lake Balkhash and Kashmir (Ḥudūd, p. 282).

The Kimāk was another Turkish nation, having its main territory in western Siberia north of the Irtish river (Ḥudūd, pp. 304–311).

Al-Rūs, from Rus', whence Russia, the name of Scandinavian freebooters and traders who set up principalities along the Russian river routes to Byzantium (Ḥudūd, pp. 432–8).

Al-Ṣaqāliba are the Slavs, of which two groups were known to the Muslim geographers, those of Macedon, and others who live near the Rūs, or who include the Rūs (Ḥudūd, pp. 427–432).

In the seventh keshvar:

Khotan is in Sinkiang province of China, north of Kashmir (Ḥudūd, p. 261).

43. The Arctic and the Tropical Regions (135:16 – 138:12)

In Marvazī (p. 34) is an account of the northernmost inhabited region which recital closely resembles that of Bīrūnī in 137:11 – 138:3. The Bulghār here named were a people living in the Volga valley. The Ayswā (or Īsū, or Wīsū, or Ves) are taken to be a community of Finns inhabiting the region near the Belo-ozero and east of Lake Ladoga. Yūra was apparently a place still farther west. In connection with Bīrūnī's mention of skis (or skates), we remark that Soviet archeologists excavating the waterlogged accumulations under Novgorod (not far from the presumed Yūra) have recovered medieval skis (Marvazī, pp. 112–4; Ḥudūd, pp. 435, 438; Thompson, pp. 82, 99).

Lank, or Lankā, is the island of Ceylon (Ḥudūd, p. 138).

Zanj, see Section 41.

Dībajāt, the Laccadive and Maldive Islands (Ḥudūd, p. 244).

Wāqwāq of the east is Madagascar, that of the west Sumatra (EI, vol. 4, pp. 1105–9).

Zābij, see Section 2.

44. The Table of Climate Bounds & Its Calculation (138:13 – 141)

The seven "climates" (cf. Honigmann; Geogr. 1, 23, ed. of Müller, vol. 1, pp. 56–59) of classical and medieval antiquity are contiguous bands on the northern hemisphere parallel to the equator, frequently defined as described below. The "middle" of the first climate is the parallel of latitude along which the maximum length of daylight is thirteen hours. The middle of the second climate has maximum daylight length of 13 1/2 hours, and so on, the increment for each successive climate being a half hour (cf. Table 1 below, column A). The beginning of the first climate is the parallel of $12{:}45^h$ maximum daylight; the end of the first climate (and the beginning of the second) is the parallel of $13;15^h$, and so on, taking increments of a quarter hour on either side of each middle.

The set of φ's so defined is the last column, G, in the table. The columns intervening between A and G are the results of successive steps in the computation, the method for which is explained in the text.

To obtain the entries of column B, the equation of (half the) daylight, simply put

$$((\text{A}/2) - 6)15^\circ = q.$$

Column C is Sin q, and column D is $\text{C} \cdot \text{Cos}\ \varepsilon / R$. Column E is $(\text{Sin}^2 \varepsilon + \text{D}^2)^{\frac{1}{2}} = \text{Sin}\ r$, where r is the rising amplitude (cf. Figure C22).

Column F is $\text{D} / \text{E} = \text{Cos}\ \varepsilon \cdot \text{Sin}\ q / \text{Sin}\ r = \text{Sin}\ \varphi$, and column G is arc Sin F, the required φ. Note that the equation just above is expression (5) in Section 39, already derived in 133:6 in the text, provided that δ is replaced by ε.

Nevertheless the author gives another derivation, this time with the aid of a plane figure (cf. Figure C24). He states that (139:9) $HZ = \text{Sin}_{HD}\ q$. This is equivalent to saying

$$\frac{\text{Sin}_{HD}\ q}{\text{Sin}_R\ q} = \frac{HD}{R},$$

or

$$\text{Sin}_{HD}\ q = HZ = \frac{HD \cdot \text{Sin}\ q}{R} = \text{Sin}\ q \cdot \text{Cos}\ \varepsilon / R,$$

which is column D. Column E is $ZE = (\overline{HZ}^2 + \text{Sin}^2 \varepsilon)^{\frac{1}{2}} = \text{Sin}\ r$, and from the similar triangles EHZ and TEK (139:16)

$$\frac{ZE\ (= \text{Sin}\ r)}{HZ} = \frac{ET\ (= R)}{TK\ (= \text{Sin}\ \varphi)},$$

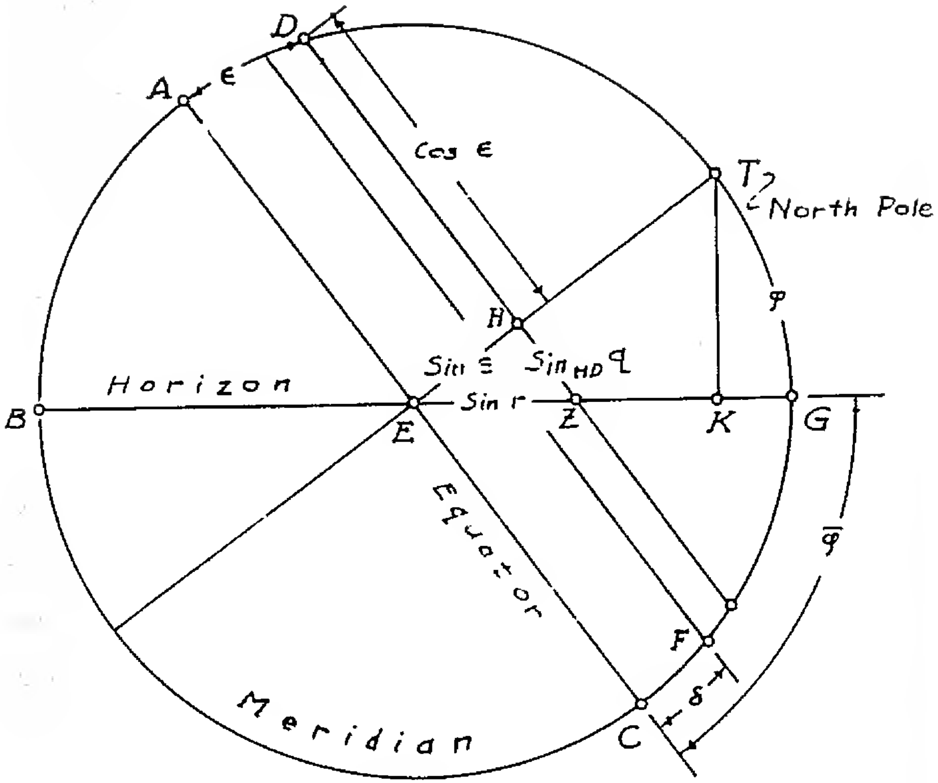


Figure C24

or

$$\text{Sin } \varphi = \overline{HZ} \cdot R / \text{Sin } r = (\text{Sin } q \cdot \text{Cos } \varepsilon / R)(R/\text{Sin } r)$$
$$= \text{Cos } \varepsilon \cdot \text{Sin } q / \text{Sin } r,$$

which is the desired relation.

Unfortunately for the accuracy of the table, two errors at the outset of the computation affect all the final results. Bīrūnī claims (140:1) that

$\text{Sin } \varepsilon =$ 24;0,57,38, and $\text{Cos } \varepsilon =$ 54;59,59,5,

whereas

Sin 23;35° = 24;0,17,48,38, and Cos 23;35° = 54;59,19,29,45.

For the sine, an inadvertent dot over the third digit in Bīrūnī's table has converted a ya' (for ten, hence 17) into a nūn (for fifty, hence 57). The same sort of thing has happened with the cosine. The poor reading of the sine goes back to Bīrūnī himself -- it cannot be a copyist's error, for if we round off the sine to 24;0,58 and square we obtain 7,474,985,764 fourths, which also appears in the text.

In Table 1 below Bīrūnī's climate bounds have been recomputed with the IBM 1620 at the American University of Beirut. The calculations have been carried to four significant sexagesimal digits -- one more than the text. Where digits in the text differ from the accurate values, or where terminal digits have been truncated when they should have been rounded, the digits of the text have been entered by hand above the corresponding machine results.

TABLE 1: RECALCULATION OF BIRUNI'S CLIMATE BOUNDS

Middle of Climate	Ⓐ Maximum Daylight	Ⓑ Equation of Daylight	Ⓒ Sine of Daylight Equation	Ⓓ Transformed Sine of the Equation	Ⓔ Sine of Rising Amplitude	Ⓕ Sine of Latitude	Ⓖ Latitude of the Climate Middle
.5	12;45	5;37, 30	5;52, 51, 42	28 5;23, 23, 25	39 24;36, 9, 21	13 13; 8, 40, 25	5 12;39, 17, 10
1.0	13; 0	7;30, 0	7;49, 53, 39	44 7;10, 38, 54	58 25; 3, 18, 2	3 17;11, 17, 12	34 16;38, 47, 56
1.5	13;15	9;22, 30	9;46, 25, 25	33 8;57, 26, 42	58 25;37, 18, 13	16 20;58, 34, 8	29 20;27, 46, 48
2.0	13;30	11;15, 0	11;42, 19, 31	48 10;43, 39, 59	18;13 26;17, 34, 51	33 24;28, 49, 50	30 24; 4, 47, 15
2.5	13;45	13; 7, 30	36,58 13;37, 28, 28	28,53 12;29, 11, 54	58 27; 3, 30, 1	40,8 27;41, 17, 45	27,40 27;28, 55, 28
3.0	14; 0	15; 0, 0	15;31, 44, 55	14,6 14;13, 55, 41	52,2 27;54, 24, 35	39 30;35, 57, 18	27 30;39, 46, 48
3.5	14;15	16;52, 30	17;25, 1, 29	56 15;57, 44, 36	50,19 28;49, 39, 40	1 33;13, 22, 58	39;36,56 33;37, 20, 42
4.0	14;30	18;45, 0	19;17, 10, 55	45 17;40, 32, 0	49,18 29;48, 37, 36	11 35;34, 33, 13	29 36;21, 55, 8
4.5	14;45	20;37, 30	21; 8, 6, 1	25 19;22, 11, 15	51,23 ·30;50, 42, 43	20 37;40, 41, 2	53,36 38;54, 1, 15
5.0	15; 0	22;30, 0	22;57, 39, 37	52 21; 2, 35, 49	56,3 31;55, 21, 38	32,45 39;33, 6, 13	13,52 41;14, 18, 45
5.5	15;15	24;22, 30	44 24;45, 44, 43	55 22;41, 39, 17	44 33; 2, 3, 33	12,49 41;13, 9, 49	5 43;23, 32, 18
6.0	15;30	26;15, 0	26;32, 14, 21	33 24;19, 15, 15	11,1 34;10, 20, 9	41,55 42;42, 10, 22	8 45;22, 28, 32
6.5	15;45	28; 7, 30	1 28;17, 1, 42	36 25;55, 17, 28	20,27 35;19, 45, 38	1 44; 1, 21, 30	26 47;11, 54, 11
7.0	16; 0	30; 0, 0	30; 0, 0, 0	30,0 27;29, 39, 45	31,30 36;29, 56, 27	41 45;11, 50, 49	21 48;52, 34, 30
7.5	16;15	31;52, 30	2 31;41, 2, 39	37 29; 2, 16, 3	41,13 37;40, 31, 12	15,13 46;14, 39, 17	24,34 50;25, 12, 34

45. The Surrounding Sea (142:1 – 145:10)

This is a discussion of the reasons why the inhabited part of the globe is no larger than it is. To north and south are extremes of cold and heat, while the sea is to east and west.

In this connection, it seems to have been Bīrūnī who introduced the term Varangian (142:12) into Muslim geography. The word was used by the Slavs and Byzantines to denote Scandinavians. The information in this passage and 137:10 – 138:3 apparently was supplied by an ambassador from the Volga Bulghārs who presented himself at Maḥmūd's court and who was interviewed by our author (Marvazī, pp. 115–6; Comm. Vol., pp. 234–5).

Farther along (at 143:10) for "the promontory called Barāsūn", read Rāsūn. Variants of the word occur elsewhere in the Islamic geographical literature. It is fearfully corrupted from a place-name in Ptolemy's Geography (Ḥudūd, pp. (64, 473–4).

The Pillars of Hercules are the two promontories standing on either side of the straits of Gibraltar. According to the legend, they were erected by Hercules upon his successful return after fetching the cattle of Gergon. Variants state either that he opened the straits, or that he narrowed them, and the pillars commemorate the act, or that he did neither and the columns simply mark his passage. But there is no question of a bridge in the legend. Nor does Ptolemy (Geogr. II, 4) say anything about a bridge.

Sūs al-Aqṣa (144:4) was the name given the westernmost part of Morocco (Ḥudūd, p. 108).

Ḥamza al-Iṣfahānī (893–935) was a prominent historian and philologist of Isfahan and Baghdad (GAL, SI, p. 221).

Vessels with sewn hulls have been used in the Persian Gulf, the Red Sea, the Indian Ocean, and off the east coast of Africa since ancient times. The notion that this is done because natural magnets would extract iron nails is also ancient. In fact, under certain circumstances this flexible mode of construction produces craft more seaworthy than the use of nails (Hourani, pp. 89–99; Newberry).

Bīrūnī's story about the finding of wreckage with stitched planks (144:14) occurs elsewhere in the literature (Reinaud, p. 90), and with the same general object of demonstrating communication between the Indian Ocean and other seas. But whereas the Taḥdīd reports planks found in the Atlantic near the Mediterranean, the other legend has them appearing in the Mediterranean itself, having floated into the Caspian (sic), thence through the Dardanelles, thus proving that the Black Sea and the Caspian both are joined to the circumambient sea.

46. Daylight and Night in the Polar Regions (145:11 – 146:12)

The northern bound of Climate 7 is $\varphi = 50;25^{\circ}$, while the beginning of the arctic zone is at $\varphi = \bar{\varepsilon} = 66;25^{\circ}$. Hence, there is, as Bīrūnī implies (145:12), a considerable gap between the two regions.

He gives as a condition for a twenty-four hour day, that the inequality $\delta > \bar{\varphi}$ must be satisfied. That this is valid can be seen from Figure C24, where the straight line extending from F represents in projection the day-circle when the sun's declination is δ. Rotate the whole configuration counterclockwise, except that the horizon is to be held fixed, until F rises above G. This corresponds to northward travel on the part of the observer, and insures twenty-four hour daylight for that δ. But then $CF > CG$, which is the condition above.

For a fixed $\varphi > \bar{\varepsilon}$, to calculate the length of this "day", put $\delta = \bar{\varphi}$. Then find from a table of solar declinations, in the part where $\delta(\lambda)$ is increasing (i.e. $0 < \lambda < 90^{\circ}$), the λ corresponding to this δ. For this value of λ regarded as solar true longitude, calculate the corresponding solar mean longitude, $\bar{\lambda}_1$. In like manner compute $\bar{\lambda}_2$ for the instant when passes through $\bar{\varphi}$ while decreasing. Then $(\bar{\lambda}_2 - \bar{\lambda}_1)/\dot{\bar{\lambda}}$ is the duration of the "day", expressed in the time units of the $\dot{\bar{\lambda}}$.

The figure invoked by Bīrūnī in the case of an imaginary observer at the pole, where he likens the spinning celestial sphere to a mill, is graphic indeed to anyone who has watched a water-driven millstone rotating about a vertical axis.

47. The Methods of Ibn al-Ṣabbāḥ and Abū Naṣr Manṣūr for Finding the Obliquity of the Ecliptic (146:13 – 155:7)

Let P be the point on the eastern horizon crossed by the sun as it rises on a day of an equinox (when $\delta = \varepsilon$). Consider a small circle on the celestial sphere having the east point on the horizon as pole and passing through P. This was known as the <u>rising amplitude circle</u>, and it has the following curious property. In general, the rising sun will cross the horizon at a point inside this circle. Project this point vertically upon the rising amplitude circle, upward if $\lambda < 180^{\circ}$, otherwise downward, and call the projection Q. Then the arc PQ at all times equals λ, the solar longitude, and Q moves along the rising amplitude circle with the same angular velocity with which the sun moves along the ecliptic. This fact seems to have been well known to the medieval Islamic astronomers, for they applied it freely without bothering to prove it. Proofs will be found in <u>Sharkas</u>.

A certain astronomer named Muḥammad ibn al-Ṣabbāḥ (about whom practically nothing is known save that he was influenced by Indian astronomy, <u>Suter</u>, p. 19) utilized this circle in a technique for determining ε. His work has not survived, but it was known to Bīrūnī's teacher, Abū Naṣr, whose description of the method is extant. Bīrūnī himself mentions it in at least two other places, including the <u>Canon</u> (p. 366, see also <u>Sharkas</u>; <u>Suter</u>, p. 81; <u>Eine</u> vol. 3, p. 808). In brief, the procedure is to observe the rising amplitude at a place of known φ on three occasions (during a single season) separated by equal time intervals. With these data the maximum rising amplitude is computed. Then, by application of the rule derived in 132:4, ε can be calculated in terms of φ and r_{max}.

The procedure for calculating r_{max} is as follows (146:19):

Put
$$2 \operatorname{Sin} r_1 = n_1$$
$$2 \operatorname{Sin} r_2 = n_2$$
$$2 \operatorname{Sin} r_3 = n_3$$

From these calculate the "extracted chord"
$$c = (n_2^2 - n_1 \cdot n_3)^{\frac{1}{2}},$$
and the "perpendicular"
$$p = \left\{ n_2^2 - \left(\frac{n_1\ n_3}{2}\right)^2 \right\}^{\frac{1}{2}}$$
Then
$$r_{max} = \operatorname{arc} \operatorname{Sin} \left\{ \frac{1}{2} \frac{c n_2}{p} \right\}$$

These operations will be justified in the course of explaining the worked example below.

If the observer is on the equator, i.e. if $\varphi = 0$, the rising amplitude each day is the solar declination at sunrise, and the maximum rising amplitude circle shrinks into a circle with polar distance ε, Bīrūnī's "declination" circle. However, this is a special case of the more general situation described above and the method is still valid. Bīrūnī takes advantage of this, and the fact that declinations (at meridian transit) are easy to observe whereas

rising amplitudes are difficult, to apply this technique to three declination observations. They were made at thirty-day intervals during the year 1016:

(149:6) Wednesday, 11 July,

$\delta_3 = 21;28^{\circ}$, $n_3 = 2 \text{ Sin } \delta_3 = 43;54,55 = 158,095$ seconds,

(149:8) Friday, 10 August,

$\delta_2 = 14;0^{\circ}$, $n_2 = 2 \text{ Sin } \delta_2 = 29;1,\overline{50} = 104,510$ seconds,

(150:2) Sunday, 9 September,

$\delta_1 = 3;12^{\circ}$, $n_1 = 2 \text{ Sin } \delta_1 = 6;41,55 = 24,115$ seconds,

where the r's have become δ's.

The proportion given at 148:10,

$$\frac{\text{Sin } \lambda}{\text{Sin } \delta(\lambda)} = \frac{R}{\text{Sin } \varepsilon} \tag{8}$$

is a direct consequence of the sine theorem applied to a spherical right triangle having hypotenuse of length λ (an arc of the ecliptic starting from the vernal point), an acute angle of ε, and a leg opposite the latter. This also establishes the fact that the sun's travel on the ecliptic is matched by equal angular travel of the corresponding point on the declination circle. This is shown on Figure 25, where the outer circle can be regarded as the ecliptic and the inner arc as a portion of the declination circle. Here they are displayed concentric and coplanar, whereas in space they are in difference planes. But since we may regard T as the sun, the arc AT as λ, and TE as R, then TF = Sin λ. If HE = Sin ε, then by similar triangles

$$TF / SO = R/SE,$$

or

$$\frac{\text{Sin } \lambda}{SO} = \frac{R}{\text{Sin } \varepsilon} .$$

If this is compared with expression (8) above, it is clear that SO = Sin δ and arc AC = δ. Motion of the sun T and its image S are thus synchronised.

We proceed to the joint calculation and demonstration, making use of Figure C26 which shows the declination circle with three points on it, B, G, and D, corresponding to the three observations. That is, A is the point at which $\delta = \lambda = 0$, and on this circle the chord of an arc twice arc AB is n_1, the chord of twice AG is n_2, and the chord of twice AD is n_3. The arcs BG = GD = $\Delta\lambda$, chord DE = 2 Sin δ_2 = n_2. Take DZ = DE and MZ = BD. Then DM = AE, and BZ = 2 Sin δ_3 = n_3.

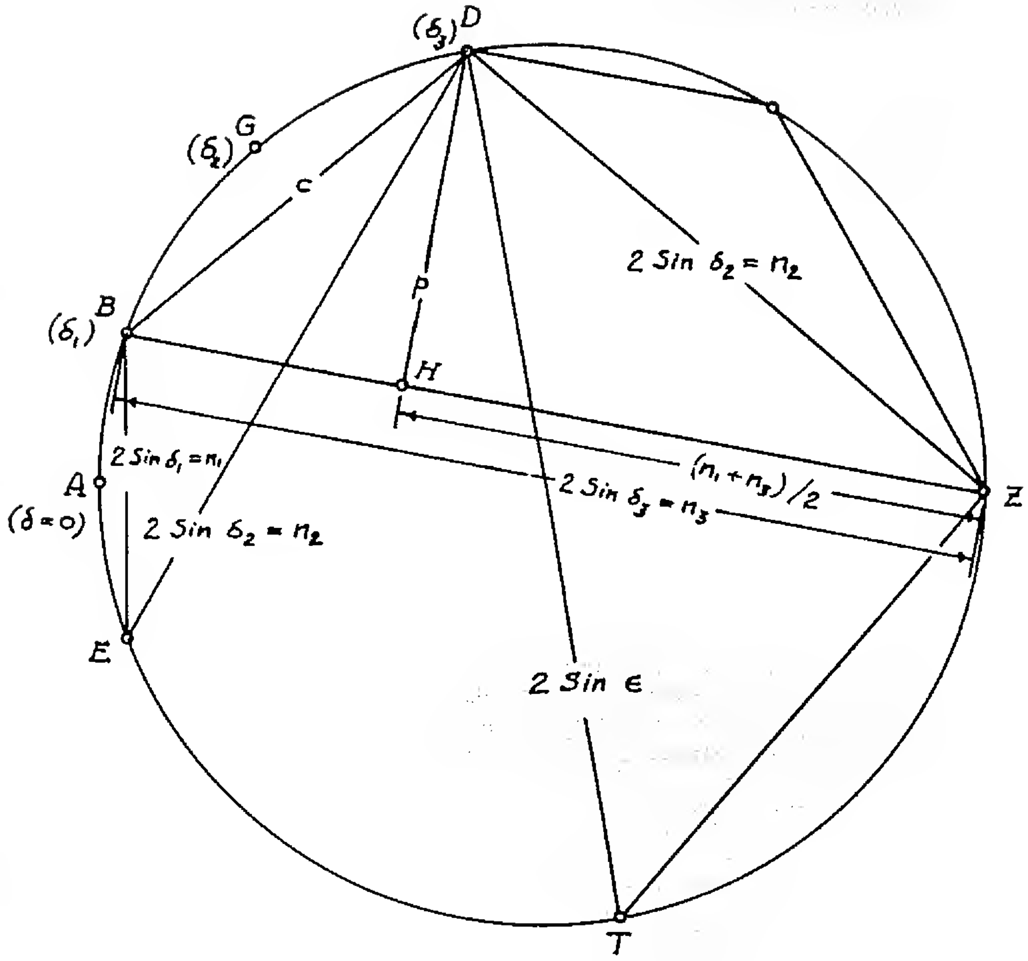


Figure C26

Now apply to the isosceles trapezoid BDMZ the theorem of Ptolemy (Almagest I, 10; vol. I, p. 28) which states that in any inscriptible quadrilateral the product of the diagonals equals the sum of the products of the two pairs of opposite sides. So

$$\overline{BM}^2 = \overline{BD}^2 + \overline{ZB} \cdot \overline{MD},$$

or

$$\overline{BD}^2 = n_2^2 - n_1 \cdot n_3 = c^2 ,$$

equivalent to the first step of the algorism for r_{max} and which gives a geometric interpretation for c. Here (151:12)

$$\begin{aligned} c^2 &= (29;1,50)^2 - (6;41,55)(43;54,55) \\ &= 10,940,340,100 \text{ fourths} - 3,812,460,925 \text{ fourths} \\ &= 7,127,879,175 \text{ fourths.} \end{aligned}$$

However, Abu Rayḥān has made an error in squaring n_2. His result should have been

$$n_2^2 = 10,922,340,100 \text{ fourths},$$

hence he would have obtained

$$c^2 = 7,109,879,175 \text{ fourths},$$

which would have given

$$c = 84,320 \text{ seconds, instead of his}$$

(151:16) 84,427 seconds.

For the second stage in the computation, we note that on the figure, if DH is drawn perpendicular to DM and BZ, HZ will be $(n_1 + n_3)/2 = 25;18,25$ the arithmetic mean of the parallel sides of the trapezoid BDMZ. Hence

$$\overline{DH}^2 = \overline{DZ}^2 - \overline{HZ}^2 = n_2^2 - \frac{(n_1 + n_3)^2}{2} ,$$

and DH is p, which is indeed a perpendicular. So (151:18)

$$\begin{aligned} p &= \left\{ (29;1,50)^2 - (25;18,25)^2 \right\}^{\frac{1}{2}} \\ &= (10,940,340,100 \text{ fourths} - 8,300,121,025 \text{ fourths})^{\frac{1}{2}} \\ &= (2,640,219,075 \text{ fourths})^{\frac{1}{2}} = 51,383 \text{ seconds.} \end{aligned}$$

But again the erroneous value of n_2^2 has been used. p^2 should have been 2,622,219,075 fourths and the correct p is 51,208 seconds. Finally, DT, the diameter of the circle through D forms a right triangle with Z, which triangle is similar to BDH. So

$$\frac{DT}{DZ} = \frac{BD}{DH} , \quad \text{or} \quad \frac{DT}{n_2} = \frac{c}{p} .$$

Therefore DT = 2 Sin ε , and (152:6)

$$\begin{aligned} \varepsilon &= \text{arc Sin} \tfrac{1}{2} \left(\frac{n_2 \cdot c}{p} \right) = \text{arc Crd} \left(\frac{104,510 \text{ seconds} \times 84,427 \text{ seconds}}{51,383 \text{ seconds}} \right) \\ &= \tfrac{1}{2} \text{arc Crd} (171,720 \text{ seconds}) = \text{arc Sin} \left[\tfrac{1}{2} (47;42) \right] \\ &= 23;25,19^\circ \end{aligned}$$

However, the author's c and p are both wrong. He should have had

$$\begin{aligned} \varepsilon &= \text{arc Sin} \left[\tfrac{1}{2} \left(\frac{104,510 \text{ seconds} \times 84,320 \text{ seconds}}{51,208 \text{ seconds}} \right) \right] \\ &= \text{arc Sin} \left[\tfrac{1}{2} (172,088 \text{ seconds}) \right] \\ &= \text{arc Sin } 23;54,4 = 23;28,30^\circ . \end{aligned}$$

Bīrūnī's criticism of the method (152:8) is valid as far as it goes. It is true that because of the solar equation, equal time intervals do not in general correspond to equal solar arcs along the ecliptic, and this is a tacit assumption of the procedure. But beyond this, successive meridian passages of the sun mark out local apparent days, not mean days. In order to compensate for this, and for the solar equation, a correction should have been introduced due to the equation of time.

Abū Naṣr's method employs only two observations of rising amplitude, but it is necessary to compute $\Delta\lambda$, the true motion of the sun in the interval between the observations. As with the algorism of Ibn al-Ṣabbāḥ we give the procedure first, and demonstrate its validity in the course of commenting upon the numerical example.

It is of interest to learn (153:4) that the method was also explained in Abū Naṣr's zīj, no longer extant, the Royal Almagest. We have it directly in a shorter treatise by Abū Naṣr (cf. Sharkas).

The computation proceeds as follows (153:6). From two observed rising amplitudes, r_1 and r_2 form

$$2 \text{ Sin } r_1 = n_1 \quad \text{and} \quad 2 \text{ Sin } r_2 = n_2 .$$

Then calculate

$$\left\{\frac{n_1 + n_2}{2}\right\} \left\{\frac{R}{2 \text{ Cos } (\Delta\lambda / 2)}\right\} = s, \text{ say. Now put}$$

$$(s^2 - n_1 n_2)^{\frac{1}{2}} = w, \text{ whence, finally,}$$

$$\frac{w \cdot 2R}{2 \text{ Sin } (\Delta\lambda / 2)} = 2 \text{ Sin } r_{max} .$$

The fact that $\Delta\lambda$ must be halved before looking up the cosine and sine is not stated in the text, but Bīrūnī does it when carrying through the computation, and Abū Naṣr explicitly prescribes it.

In his worked example Bīrūnī again uses the results of his declination observations at Jurjāniya in 1016, those of 10 August and 9 September. Hence all r's become δ's in the expressions above, and r_{max} becomes ε. Again (153:16)

$$n_1 = 6;41,55 = 24,115 \text{ seconds}$$

and $$n_2 = 29;1,50 = 104,510 \text{ seconds.}$$

On the declination circle of Figure C27, points A, B, and D correspond to declinations of 0^o, δ_1, and δ_2 respectively. Hence, by the property of the declination circle derived above, arc BD = $\Delta\lambda$, which the author calculates as (154:5) $29;17^o$. Hence, $\Delta\lambda / 2 = 14;38,30^o$ (154:12); its sine is 15;9,59 = 54,599 seconds (155:1), and its cosine 58;3,5. (In the English translation Figure 27 is somewhat misleading because in it H falls on FO. In general, this will not be the case.)

In the isosceles trapezoid BDCZ the bases are n_1 and n_2, and since DH is normal to both, HZ is their arithmetic mean, 17;51,32 (153:18) according to the text. This number is in fact 17;51,52. In the right triangle HDZ (154:13)

$$HZ / DZ = \text{Cos } Z / R ,$$

or

$$DZ = \frac{HZ \cdot R}{\text{Cos } Z} = \frac{n_1 + n_2}{2} \quad \frac{R}{\text{Cos } (\Delta\lambda / 2)}$$

$$= \frac{17;51;32}{58;3,5} = 18;25,55 = s ,$$

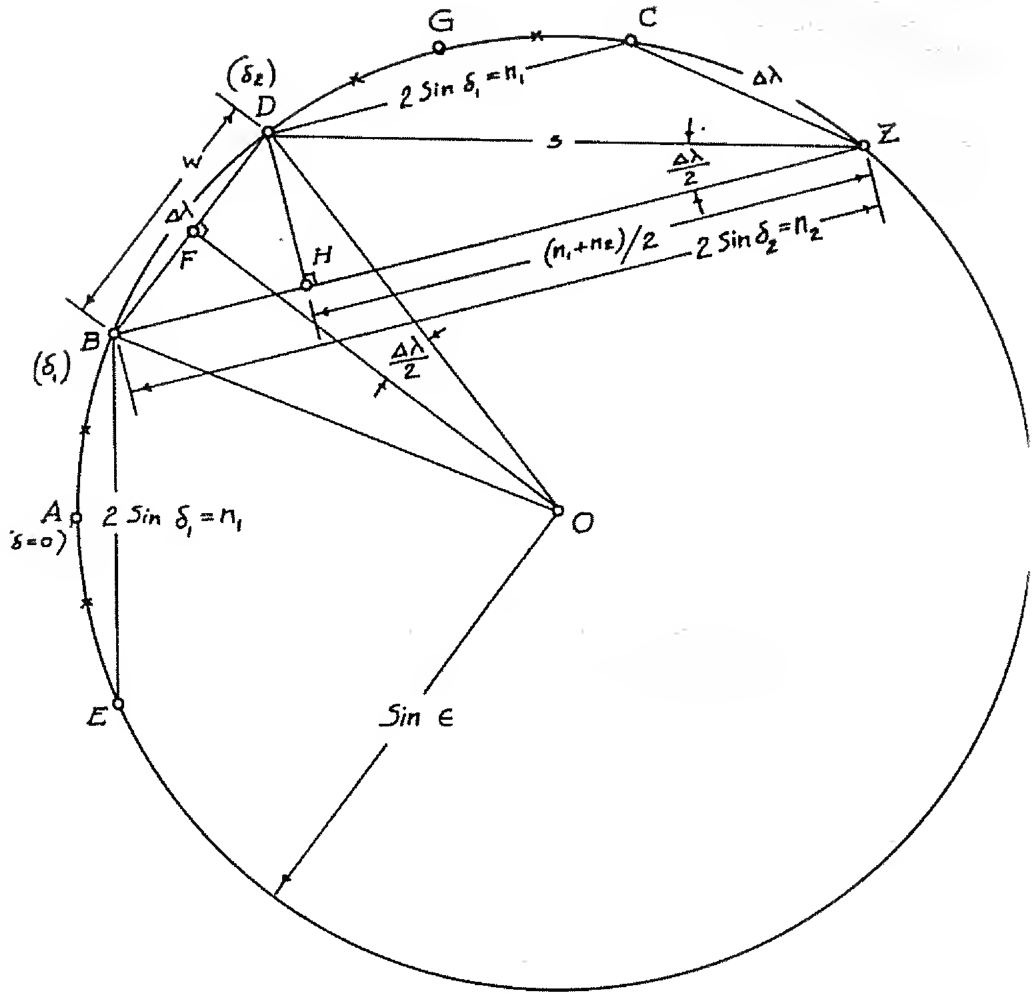


Figure C27

according to the text (154:17). There is a mistake in the division; the correct quotient of the above is 18;27,30. But here the wrong value for HZ has been used. The expression should have been $s = 17,51;52 / 58;3,5 = 18;27,51$.

By the theorem of Ptolemy applied above, in quadrilateral BDCZ,

$$\overline{BD}^2 = \overline{DZ}^2 - \overline{BZ} \cdot \overline{BC}$$

so (154:20)

$$BD = (s^2 - n_1 n_2)^{\frac{1}{2}} = (4,402,986,025 \text{ fourths} - 2,520,258,650 \text{ fourths})^{\frac{1}{2}}$$

$$= (1,882,727,375 \text{ fourths})^{\frac{1}{2}} = 43,390 \text{ seconds} = w.$$

Had the correct value of s been used, the result would have been

$$w = (4,418,393,025 \text{ fourths} - 2,520,258,650 \text{ fourths})^{\frac{1}{2}}$$

$$= (1,898,135,191 \text{ fourths})^{\frac{1}{2}} = 43,568 \text{ seconds}.$$

Since OBD is an isosceles triangle, OF, the altitude to the base, bisects the angle at O. Hence angle FOD $= \Delta\lambda / 2$. So

$$DF / DO = \text{Sin}(\Delta\lambda / 2) / R,$$

and (154:21)

$$DO = DF \cdot R / \text{Sin}(\Delta\lambda / 2) = \text{Sin}\ \varepsilon .$$

Now $DF = w/2 = 21,695$ seconds according to the text, but the correct value is 21,784 seconds. From this,

$$\text{Sin}\ \varepsilon = 21,695 \times 60 / 54,599 = 1,301,700 / 54,599 = 23;50,28,$$

according to the text. But the correct computation is

$$\text{Sin}\ \varepsilon = 21,784 \times 60 / 54,599 = 1,307,040 / 54,599 = 23;56,20.$$

Bīrūnī's final result is

$$\varepsilon = 23;24,46^\circ ,$$

but the correct result is

$$\varepsilon = 23;30,51^\circ .$$

We note that this latter is only three minutes of arc off the accurate value of ε for Abū Rayḥān's time (see Section 31), which speaks well for his observational technique, if not for his computation.

CHAPTER V. ON THE DETERMINATION OF LONGITUDINAL DIFFERENCES

48. Base Meridians (156:1 – 158:8)

Bīrūnī commences the discussion of longitudes by remarking that the choice of a zero meridian is largely arbitrary, and that at his time two such choices were in common use.

People of the west, he says, regard the Canaries (the Fortunate Isles) as the westernmost inhabited regions, and hence measure longitudes east from them. This is easy to verify, for the geographical tables in several zījes state explicitly that this is the case. Some of these tables also give a longitude of 180° to the mythical Iranian castle of Kangdizh, listed as the easternmost point, which supports his statement that the inhabited regions were supposed to cover half a circumference.

However, little evidence is at hand to support his claim (also stated by Honigmann, pp. 132–155) that the peoples of the east reckoned longitude west from this eastern meridian. Of some thirty-three sets of medieval geographical tables examined by us, all place the base meridian in the west.

It is true that these tables can be divided into two classes in such fashion that for one class the longitude of a particular locality tends to differ by ten degrees from the longitude of the same locality in a table of the other class. The reason for this difference seems to have nothing to do with any oriental base meridian; it is simply that some tables state that longitudes are taken from the eastern coast of the Atlantic, whereas others are from the Canary Islands, and the latter are taken ten degrees west of the former, (Cf. Geogr. Tables, also EI, vol. iii, p. 880).

The astronomer mentioned in 157:9 was probably Muḥammad b. Ibrāhīm b. Ḥabīb al-Fazārī (d.c. 770) a prominent Abbasid scientist whose work was strongly influenced by Indian sources. His zīj (or zījes) has not survived, but two entries from the geographical table have been transmitted, and they indeed imply a base meridian in the Far East. However an entry in Bīrūnī's Canon (vol. 2, p. 547) indicates that the number given in 157:9 as 13;30° should be 10;50°. (See Fazārī, esp. pp. 115–6, Suter, pp. 3, 4.)

Bīrūnī's pessimism expressed in 158:10 is explainable in terms of his miserable situation while he was writing the Taḥdīd, alluded to in 119:4 (Section 32). He has twice previously (in 38:1–12 and 110:7–17) mentioned his zeal in geographical work and interferences with his program of studies.

For a base meridian at the "Cupola" see Section 67 below.

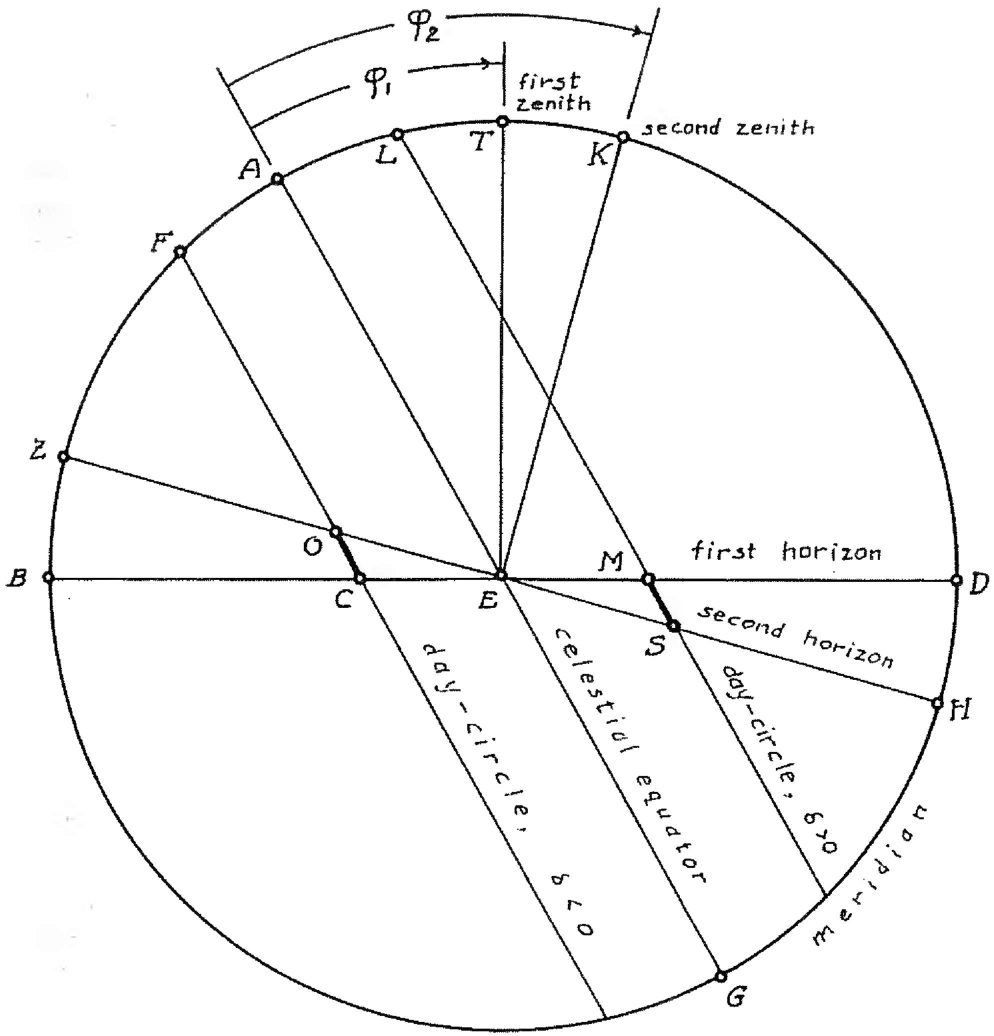


Figure C28

49. Sunrise Time, Localities on the Same Meridian (158:9–160:7)

This passage commences a discussion of the relation between time of sunrise, or daylight length, and the geographical coordinates of pairs of localities. First considered is the special case when the longitudes are the same $\Lambda_1 = \Lambda_2$, but $\varphi_1 \neq \varphi_2$. Our Figure C28 is an adaptation of the text's Figure 28 made by projecting the celestial sphere orthogonally on the common meridian ABGD. Since all the other circles involved, the horizons, day-circles, and the equator, are in planes normal to the meridian, they project as the straight lines shown. As Bīrūnī says, the difference in time of sunrise between the two localities, assuming $\varphi_1 < \varphi_2$ and $\delta > 0$, is the time required for the sun to move in its daily rotation along the day-circle arc SM. Sunrise at the second place precedes that at the first by this amount. The situation is reversed when $\delta < 0$. Then sunrise at the first place occurs first, when the sun crosses C, followed by its rising for the second locality, at O.

50. Sunrise Difference, Localities Having the Same Latitude (160:8 – 162:13)

This second special case has $\Lambda_1 \neq \Lambda_2$, but $\varphi_1 = \varphi_2 = \varphi$. The situation is portrayed on Figure C29, which is an adaptation of the text's Figure 29. The author wishes to show that the difference in time of sunrise at the two localities equals $\Delta\Lambda$, the difference in their longitudes, KD on the figure. Here, as previously, time is measured as arc distance of daily rotation along the celestial equator or any circle parallel to it. Moreover we repeatedly use the fact that the length of an arc in degrees is invariant under projection from a pole to any parallel circle.

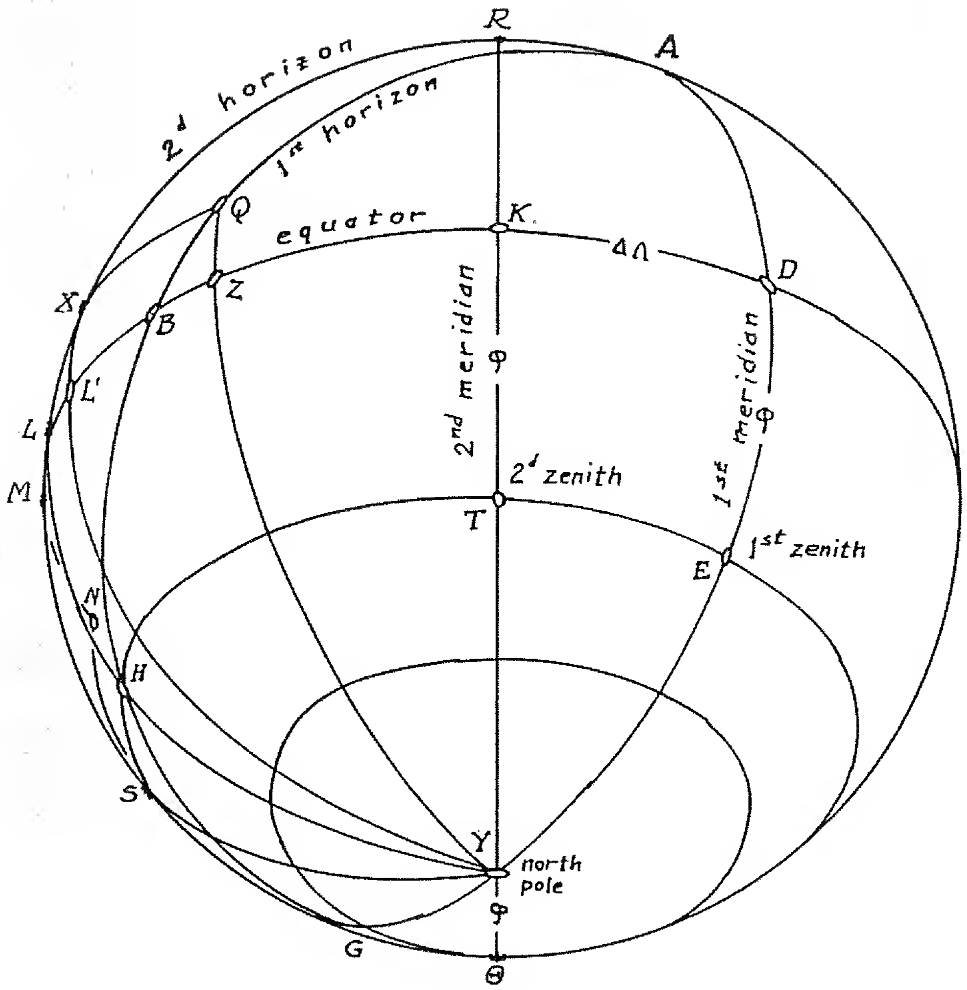


Figure C29

The method adopted is to prove the hypothesis first for the special case where the sun passes through both zeniths, T and E, and then to extend the result to cover the general case. To do the latter, Bīrūnī considers separately a daily circle, FO, made when the sun is north of the zeniths, and a second, XQ, traversed when the sun is to the south. The method of proof is the same, and to avoid further complicating the figure FO has been omitted from our figure. The proof with XQ will suffice.

Both in the text and the translation the figure is misleading. X is the rising point at the second horizon for an arbitrary day-circle. H is the rising point at the first locality for a day-circle through the zeniths. In general, these two points will not lie on the same great circle through Y, the north pole, but Figure 29 is drawn as though they did. Figure C29 shows two distinct circles, one intersecting the equator at L, the other at L . To proceed with the proof, assume first that the day-circle is SHTE. The sun crosses the first horizon at H, and the second at S. Hence arc SH is the time difference between sunrise at the two places. We wish to show that SH = TE = KO = $\Delta\Lambda$. Now BL = MN, for BL is the equation of half the daylight at the first locality, and MN is the equation of half the daylight at the second locality, and the latitudes being equal the equations of daylight are also (161:12). Also DB = KM = 90°.
So

$$DL = DB + BL = KM + MN = KN,$$

and

$$DK = DL - KL = KN - KL = LN.$$

But SH and TE have the same number of degrees as NL and KD respectively, hence

$$SH = TE = \Delta\Lambda$$

For the general case, ZB and L'M are half the daylight equations for two localities with the same day circle and the same latitude. So ZB = L'M (162:10). Hence

$$DZ = DB - ZB = 90^{\circ} - ZB = KM - L'M = KL',$$

and

$$DK = DZ - ZK = KL' - ZK = L'Z,$$

or

$$\Delta\Lambda = XQ ,$$

and XQ is the sunrise difference.

51. Sunrise Difference, Both Coordinates Different (163:1 – 165:18)

Having just shown that when $\varphi_1 = \varphi_2$ the difference in sunrise times for the two localities equals the difference in longitudes, Bīrūnī proceeds to demonstrate that this is not the case when $\varphi_1 \neq \varphi_2$ (and $\Lambda_1 \neq \Lambda_2$). In order to do so he goes through an elaborate demonstration, or rather a pair of demonstrations, in each of which the sun is assumed to pass through the zenith of one of the two localities (i.e. $\delta_s = \varphi_1$ and $\delta_s = \varphi_2$). He says nothing about the general case where δ may have any value in the range $-\varepsilon \leqslant \delta \leqslant \varepsilon$. Nevertheless his proof is redundant rather than insufficient, for to disprove a hypothesis it is sufficient to exhibit one case for which it fails to hold, and he has two.

Assume that $\varphi_2 > \varphi_1$ and put $\delta_s = \varphi_1$. Then, in Figure C30 (or 30) BL is the equation of half the daylight for the first locality, and MN for the second. By the tangent case of the Rule of Four (163:8)

$$\frac{\text{Sin BL}}{\text{R}} = \frac{\text{Tan LH}}{\text{Tan }\overline{\text{YG}}} , \quad \text{and} \quad \frac{\text{Sin MN}}{\text{R}} = \frac{\text{Tan (LH = NS)}}{\text{Tan }\overline{\text{Y}\Theta}} .$$

$$\text{Or} \quad \frac{\text{Sin BL}}{\text{Tan LH}} = \frac{\text{R}}{\text{Cot YG}} , \quad \text{and} \quad \frac{\text{Tan NS (=Tan LH)}}{\text{Sin NM}} = \frac{\text{Cot Y}\Theta}{\text{R}}$$

Multiplying these two equations gives (163:16)

$$\frac{\text{Sin BL}}{\text{Sin MN}} = \frac{\text{Cot Y}\Theta}{\text{Cot YG}}$$

Now $\quad \overline{\text{Y}\Theta} = \bar{\varphi}_2 < \bar{\varphi}_1 = \overline{\text{YG}}$.

So $\quad \text{Tan }\overline{\text{Y}\Theta}\text{ (=Cot Y}\Theta) < \text{Tan }\overline{\text{YG}}\text{ (=Cot YG)}$.

Therefore $\quad \text{Sin BL} < \text{Sin MN},$

and (164:2) $\quad \text{BL} < \text{MN}$.

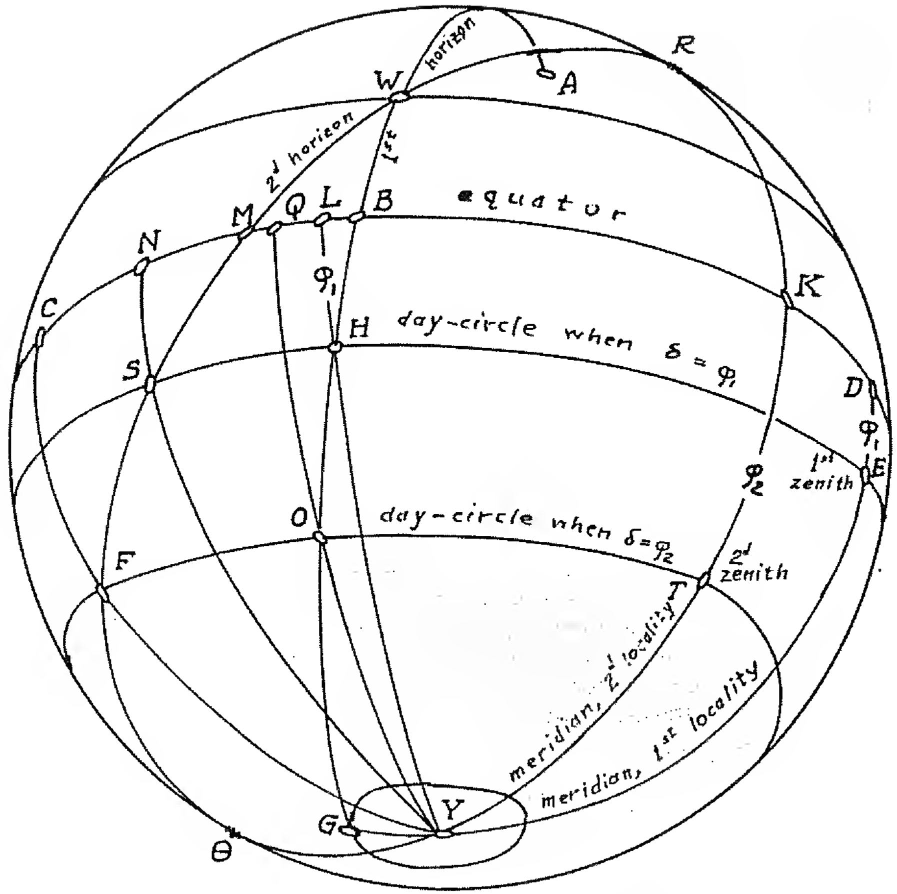


Figure C30

The point L can be thought of as marking the time of sunrise for the first locality and N for the second. Then

$$NL = MN + ML > BL + ML = MB = MD - BD = MD - 90^\circ$$

$$= (KD + 90^\circ) - 90^\circ = KD .$$

Or $NL \neq KD$, which says that the difference in sunrise times does not equal the difference in longitude, Q.E.D. This argument is equivalent to Bīrūnī's 164:2–6.

In the same manner, if the sun passes through T, the second zenith, i.e. $\delta_s = \varphi_2$, the points Q and C mark sunrise in the first and second localities respectively. It can be shown by the san sort of argument that QC, the difference in these sunrise times, is not equal to KD either.

Bīrūnī takes up briefly (165:1–4) the very special situation arisin when the day-circle passes through the point of intersection between the two horizons. As he remarks, when $\Lambda_1 = \Lambda_2$, this can happen only when $\delta_s = 0$, for the two horizons intersect at their common east point (see Figure C28). He does not deal with the general case.

The next paragraph (165:5–8) remarks that it is possible for sunrise to occur earlier at the first locality, even if it is farther west than the second. This will happen if the sun is so far south that it passes to the south of W on Figure C30. There seems to be a copyist's error in the second sentence of this paragraph. If L is replaced by W the statement makes sense.

In 166:9–18 the author gives a second example of a spherical astronomical configuration which seems to violate common sense, but which is nevertheless a fact. He states that for $\varphi = 36^\circ$, the latitude of Rhodes (Climate 4 in <u>Almagest</u> II, 13), and for

$h_s = 42^\circ$ (east)

when $\lambda_s = 324^\circ$, $\lambda_H = 69^\circ$,

but when $\lambda_s = 353^\circ$, $\lambda_H = 69^\circ$ also.

where the subscript H denotes the ascendant, or horoscope. It is natural to suppose that for a fixed solar altitude the ascendant tends to lead the sun by a more or less fixed ecliptic arc length. In this example, however, while the sun has progressed almost a zodiacal sign, the ascendant finds itself where it was before. This curiosity is easily demonstrated with an astrolabe, and no doubt it was this instrument which suggested it to Abū Rayḥān. Insert the plate for the latitude of Rhodes in the mater of the device, and rotate the rete until II 9° (=69°) on the ecliptic scale crosses the eastern horizon curve. Then it will be seen, as in Figure C30.1, (adapted from a

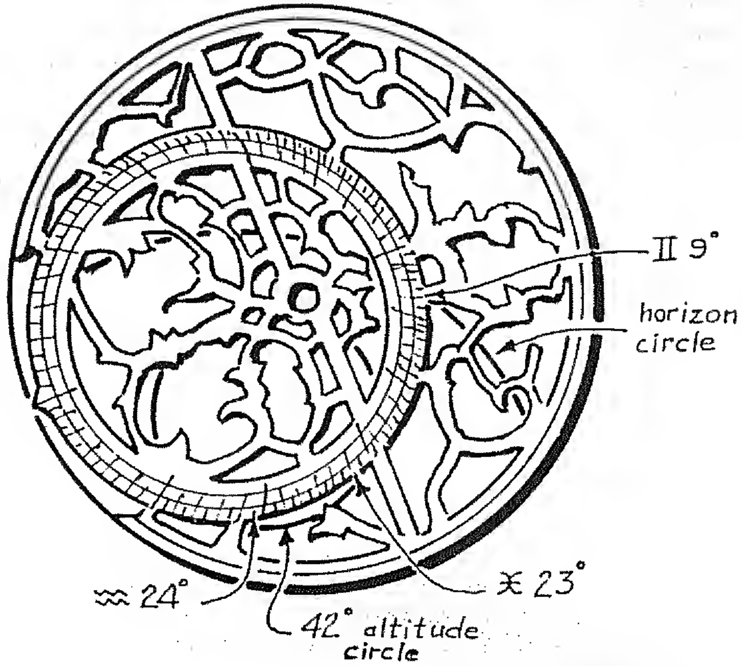


Figure C30.1

photograph by Professor O. Gingerich), that the altitude circle for 42° crosses the ecliptic scale at two points, ♒ 24° (=324°) and ♓ 23° (=353°).

Several of the treatises addressed to Bīrūnī by Abu Naṣr (165:17, see Section 47) have been published as Rasā'il Abī Naṣr ilā al-Bīrūnī, Hyderabad-Deccan, 1948. The one here referred to does not seem to be among them, however.

52. Longitude Determination from Eclipses (166:1 – 169:9)

To ascertain the longitude difference between two stations it suffices to note, in the local time of each station, the occurrence of some event. The difference in the two local times, converted into units of the daily rotation, is then the desired difference in longitude. The catch is to find an event which, like a radio signal, is simultaneously detectable at both stations. As Bīrūnī observes, if the phenomenon is a visible one, it must happen sufficiently high above the earth so that the curvature of the earth will not prevent its being seen by one observer. Also, it must not be affected by the location of the latter, which rules out lunar crescent and stellar first visibilities. The same goes for solar eclipses, for the times, phases, and magnitudes of these are strongly affected by the station from which they are observed.

This point was evidently missed by al-Hirawī (see Section 23 above). His book, al-Madkhal al-Ṣāḥibī, is not extant. Bulgakov infers (RT, p. 317, note 661) from the title that it was dedicated to the Buyid wazīr al-Ṣāḥib Ismaʿīl ʿAbbād. In like manner Abū Naṣr Manṣūr's al-Majisṭī al-Shāhī (153:4) was written in honor of the Khwārazmshāh.

The only suitable phenomenon left is a lunar eclipse, and it presents the difficulty that the instants of first contact and of clearance cannot be observed. The moon first enters the earth's penumbra, and is only gradually darkened. Some authorities claim, says Bīrūnī, that the eclipse becomes apparent only after it has reached a magnitude of one digit. This means that the shadow has obscured a twelfth of the moon's apparent diameter.

We seek to verify the numbers given in this passage, 168:3 – 10. Time measured in degrees of daily rotation equates a day, 24^h, to 360°. Hence an hour is 15°. The translation has $1;48^\circ$ of time, but the text gives $1;49^\circ$. Converting the latter into hours gives $1;49^\circ / 15 = 0;7,16^h$. The printed text and the translation have $0;6,16^h$, but the MS has $0;7,16^h$, so this is to be accepted and both numbers are secure as restored (i.e. $1;49^\circ$ and $0;7,16^h$) and are equivalent. An attempt to derive this equivalent is as follows:

A common value for the mean apparent lunar diameter is $0;32^\circ$, and one digit is a twelfth of this. The mean rate of elongation between sun and moon (or the earth's shadow and the moon) is $12;11^\circ/d$. Hence the mean time for the shadow to cover a digit is

$$\frac{0;32}{12} \div \frac{12;11}{24} = 0;5,15^h ,$$

which is considerably less than the number given in the text. There exists a second variety of eclipse digit, defined as a twelfth of the surface of the apparent disk. Almagest VI, 8, Table 4, gives conversions between such lunar areal digits and the linear digits. There an areal digit equals 1;45 linear digits. So the mean time for the shadow to cover such a digit is

$$\frac{0;32 \times 1;45}{12} \div \frac{12;11}{24} = 0;9,10^h ,$$

which exceeds the desired 0;7,16.

A "seasonal (or unequal) hour" (169:7) is defined as being a twelfth of the time from sunrise to sunset. These units vary in length with the local latitude and with the season.

53. A Digression on Combinatorial Analysis (169:10 – 170:14)

Bīrūnī enumerates seven ways in which the time of a lunar eclipse may occur or be recorded. Bearing in mind that such an eclipse requires that the sun and moon be in conjunction, they are:

1. Sunset
2. Time from sunset until the eclipse
3. Time from the eclipse until midnight
4. Midnight
5. Time from midnight until the eclipse
6. Time from the eclipse until sunrise
7. Sunrise.

He proceeds to calculate the number of ways in which the times of an eclipse can be recorded

$$(1 + 2 + 3 + \ldots + 7)\,2 = \frac{7 \times 8}{2} \times 2 = 28 \times 2 = 56.$$

We are unable to fathom his reasoning. The expression is reminiscent of the formula for the number of combinations of n things taken two at a time:

$$C_2^n = \frac{n\,(n-1)}{2} \quad \text{for} \quad C_2^8 = \frac{8 \times 7}{2} = 28.$$

But our number is seven, not eight.

At each of the stations the latitude may be either known or unknown. This quadruples the possibilities, as Bīrūnī rightly remarks, yielding $56 \times 4 = 224$. The translation has 256 (170:3).

In fact the time may be recorded in each of seven different ways at the first locality, and each of these may be combined with any one of the seven possibilities at the second locality. Hence, including the latitude combinations, the number of possibilities is

$$7 \times 7 \times 4 = 196.$$

Bīrūnī was evidently reminded of another problem, this one having nothing to do with geography or astronomy. It involves a statement which may be translated as, "He who is standing can not (at that instant) be a seated individual". This sentence was, for the logicians of the time, a famous example of the assertion that a proposition cannot be both true and false.

Yaḥyā b. ʿAdī, a Christian and well-known logician who died c. 975 in Baghdad (*Suter*, p. 59; see also Section 57 below), at one time asserted that (the letters of ?) the statement can be arranged in

$$16,384 = 2^{14}$$

different ways. Later he claimed that the number is

$$18,432 = 2^{11} \cdot 3^2 .$$

Al-Ḥasūlī, the second individual cited in connection with this problem, seems to be named nowhere else in the literature. His solution is

$$25,088 = 2^9 \cdot 7^2$$

A fourth solution is attributed to ʿĪsā b. Yaḥyā (d. 1010) a famous physician, philosopher, and mathematician who lived in Gurgān (*Suter*, p. 79; *RT*, p. 317, note 669).

In the absence of information as to precisely how the arrangements are to be made, we are unable to infer the methods which led to these numbers. Counting twice the doubled letter *nūn* in the sentence, there are eighteen letters. The number of permutations of n things taken n at a time is the factorial, $\underline{|n}$. But factorials of the order of 17 or 18 are far larger than even ʿĪsa's result. Of course, some letters are repeated. There are two *qāfs* and four *alifs*, not counting the initial *hamza*.

54. A Lunar Eclipse Observed from Two Localities, Both Times With Respect to the Local Meridians (170:15 - 174:1)

This is the beginning of a long passage, running to 185:12, which examines the multitude of special cases enumerated just above. The reason for the discussion is that the eclipse time at each locality may be measured with respect to the local midnight (i.e. the local meridian), or with respect to local sunrise (or sunset). If the latter is done, then the local equation of daylight must be calculated, which in turn implies that the local latitude must be known. Time measured from a horizon phenomen must be converted to time from the meridian. For ultimately it is the difference between the two meridians which is required. This is the crux of the remarks in 170:15 - 171:4.

Bīrūnī then commences with four cases which involve no horizon phenomena. They are for the eclipse occurring

(1) at midnight in both localities,
(2) before (or after) midnight in both,
(3) at midnight in one and before (or after) midnight in the other, and
(4) before midnight in the one and after midnight in the other.

It is legitimate to interchange "midnight" and "meridian" here, since the middle of the eclipse occurs at opposition, so that if the eclipse takes place in the meridian the sun will then be crossing the meridian below the horizon.

Provided we admit the use of positive and negative arcs, which Bīrūnī could not, all four cases may be subsumed in the single Figure C31. The two localities have zeniths at E and H as shown. The times between the eclipse and local midnight at E and H are arcs SD and SY respectively, and the desired difference in longitude is

$$\Delta\Lambda = YD = SD - YD.$$

If the eclipse is observed at midnight in one or both of the two places, the corresponding arc or arcs will be zero; if it is after midnight the corresponding arc is to be taken negative.

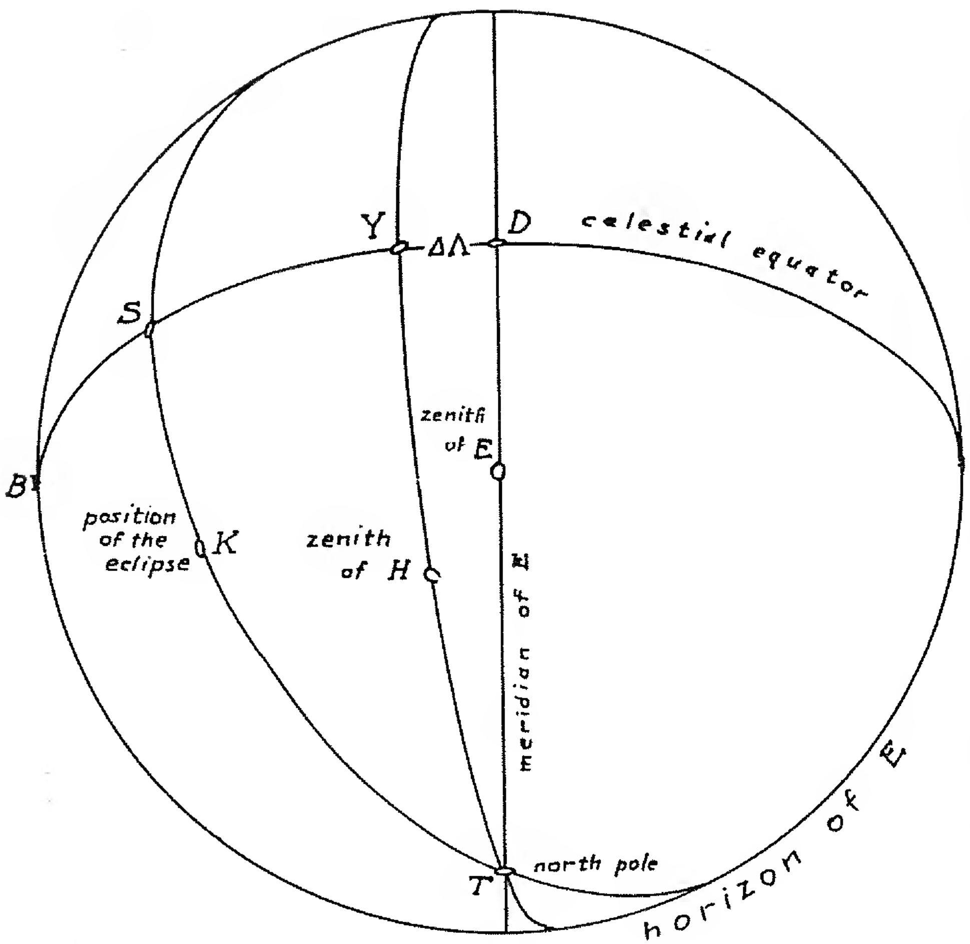


Figure C31

55. Times Observed from One Meridian and the Other Horizon (174:2 - 179:1)

This section discusses in detail the determination of longitudinal difference when the lunar eclipse time is observed with respect to the local meridian at one station but with respect to a horizon phenomenon (sunset or sunrise) at the other. In reading the passage the following facts should be borne in mind. They are not stated by Bīrūnī but are implicit in his approach.

If, in Figure C36, K marks the place on the celestial sphere where the middle of the eclipse occurs, the sun at that time will be diametrically opposite K below the horizon. Moreover, in the few hours between the eclipse and sunset (or dawn), the proper motion of the sun will amount to very little. Hence if K's declination is positive the sun's will be negative to the same amount, and the day-circles of the sun and K will be congruent parallels of declination on opposite sides of the equator. The rising of K will coincide (very nearly) with sunset; its setting with sunrise, and K's equation of daylight is the sun's equation of the night, as it were, the negative of the equation of daylight for the particular locality and season. Hence, for instance, it is legitimate to state that arc ZK is the time elapsed from sunset at E until the eclipse.

Six cases are enumerated (174:7 - 175:3). For an eclipse occurring

(1) on the meridian of one locality, with the time measured from sunset (or to sunrise) at the other.

(2) on the meridian of one, and on the horizon (east or west) of the other.

(3) with its time measured to midnight at the one locality and from sunset at the other (or after midnight at the one and to sunrise in the other).

(4) with its time measured to midnight at the one, it being on the eastern horizon at the other (or after midnight at one and on the western horizon at the other).

(5) with its time measured from sunset at the one and from midnight at the other (or the time to midnight at the one and to sunrise at the other).

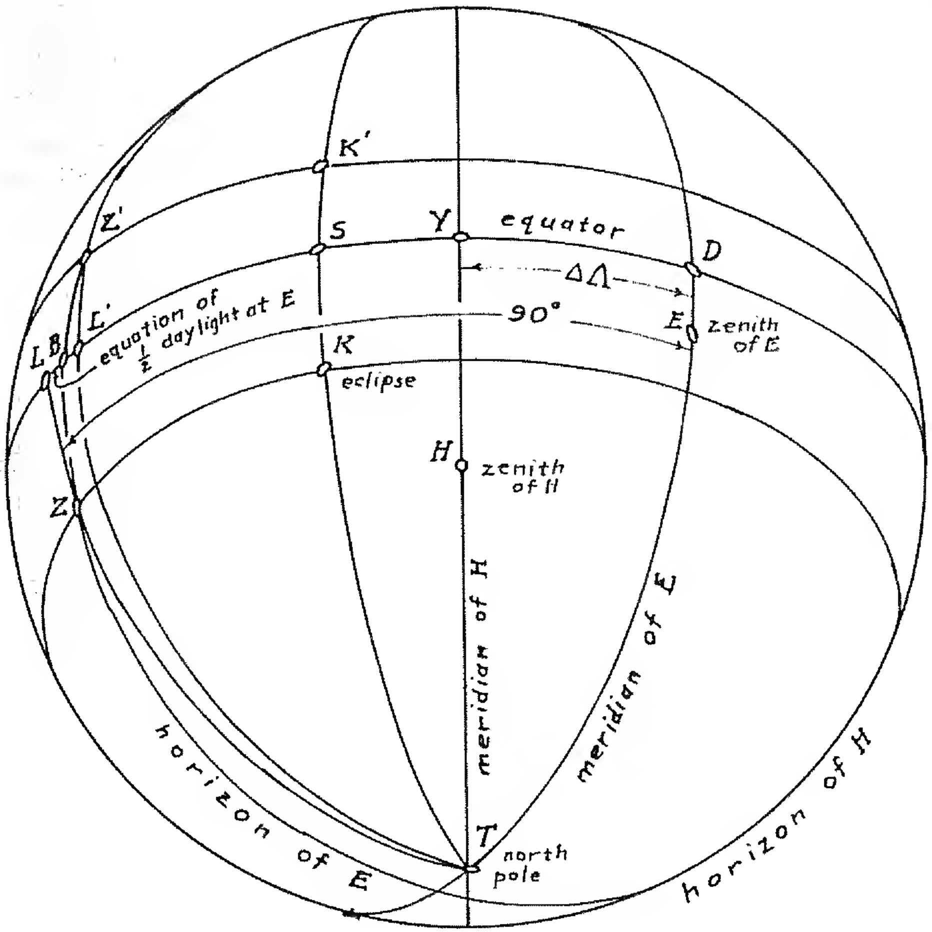


Figure C36

(6) on the eastern horizon at one and the time measured after midnight at the other (or on the western horizon at the one and the time to midnight at the other).

It will suffice to run through the procedure for case (3) (176:6–14) using Figure C36. In it arc SY is the observed time from the occurrence of the eclipse to midnight at locality H, and ZK (= LS) is the time from sunset at locality E until the eclipse. LB is the equation of half the night at E, and can be calculated if φ_E and the solar declination are known. Then the desired difference in longitude is

$$\Delta\Lambda = YD = \overline{BY} = 90^{\circ} - (BS + SY) = 90^{\circ} - \left[(LS - LB) + SY\right] .$$

If the eclipse occurs south of the equator, say at K, the equation of half the daylight, L'B, is to be taken as negative and the above procedure remains valid. For the symmetrical case (3), i.e. times after midnight and to sunrise, a figure which is the mirror image of this one will do the trick.

The other cases go through without any essential difference in approach. If the eclipse takes place on the meridian of one locality, cases (1) and (2), SY degenerates to zero. If it occurs at sunset or sunrise, cases (2), (4), and (6), LS = 0.

In Figure 38 of both the text and the translation an arc is missing. From the lower of the two Z's a parallel of declination should extend cutting the arc ST in a second K. On all the figures the repeated K's, L's and Z's show the situations for both positive and negative declinations for K.

56. Times Observed from the Two Horizons (179:2 – 185:11)

This final category of lunar eclipse times observed from pairs of localities consists of those in which the local meridian is not used at all. The cases enumerated include situations in which the times are

(1) measured after sunset (or to sunrise) at both localities.
(2) at sunset (or sunrise) at both.
(3) at sunset for one and after sunset for the other (or at sunrise for the one and time to sunrise for the other).

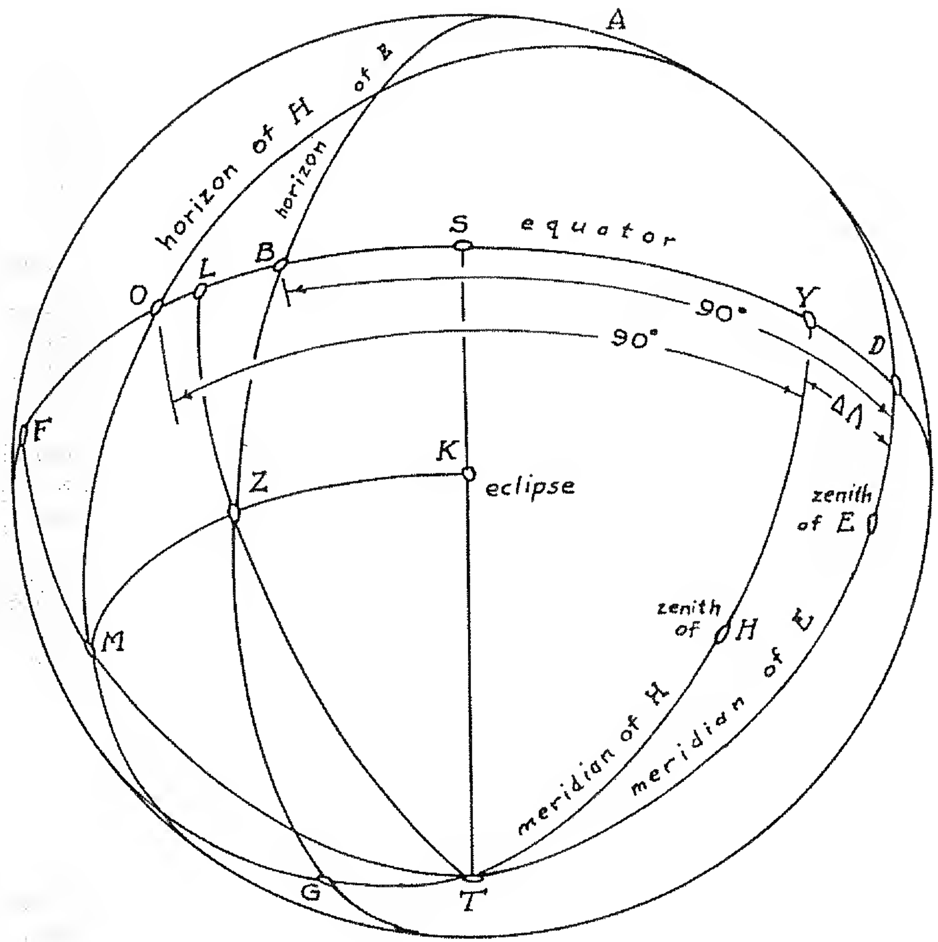


Figure C40

(4) measured after sunset at one and at sunrise for the other (or before sunrise at one and at sunset for the other).

(5) measured after sunset at one and time until sunrise at the other.

(6) at sunrise for the one and at sunset for the other.

In the above, (2), (3), (4), and (6) are special cases of (1) and (5), for whenever the sun is on one of the horizons one of the arcs in the general configuration degenerates to zero. We therefore confine ourselves to the solutions for the general cases.

For case (1) (180:1 - 5) Figure C40 has been drawn so that K, the position of the eclipse has $\delta > 0$. For $\delta < 0$ the discussion is still valid provided the customary change in signs is made. In the text and translation at 180:3 LF should be corrected to SF, ZM to KM, SF to SL, and KM to KZ. Now SF = KM is the observed time from sunset at locality H until the eclipse, and SL = KZ is the observed time from sunset at locality E until the eclipse. Also OF and BL are the equations of half daylight at H and E respectively, and can be calculated by use of φ_H and φ_E , these being assumed known: Then

$$LF = SF - SL$$

and

$$\Delta\Lambda = YD = OB = LF + LB - OF.$$

As previously, the symmetrical case where sunrise replaces sunset can be disposed of by using a figure which is the mirror image of C40.

For case (5) (183:7 - 184:7) consider Figure C44, again drawn for an eclipse position K having a northern declination. Now SF = KM is the observed time from the eclipse until sunrise at H, and SL = KZ is the observed time from sunset at E until the eclipse. Also, OF and BL are the equations of daylight at H and E respectively, to be calculated from φ_H and φ_E

$$BL - SL = BS, \qquad \overline{BS} = SD,$$

$$SF - OF = OS, \qquad \overline{OS} = SY,$$

and

$$\Delta\Lambda = YD = DS + SY .$$

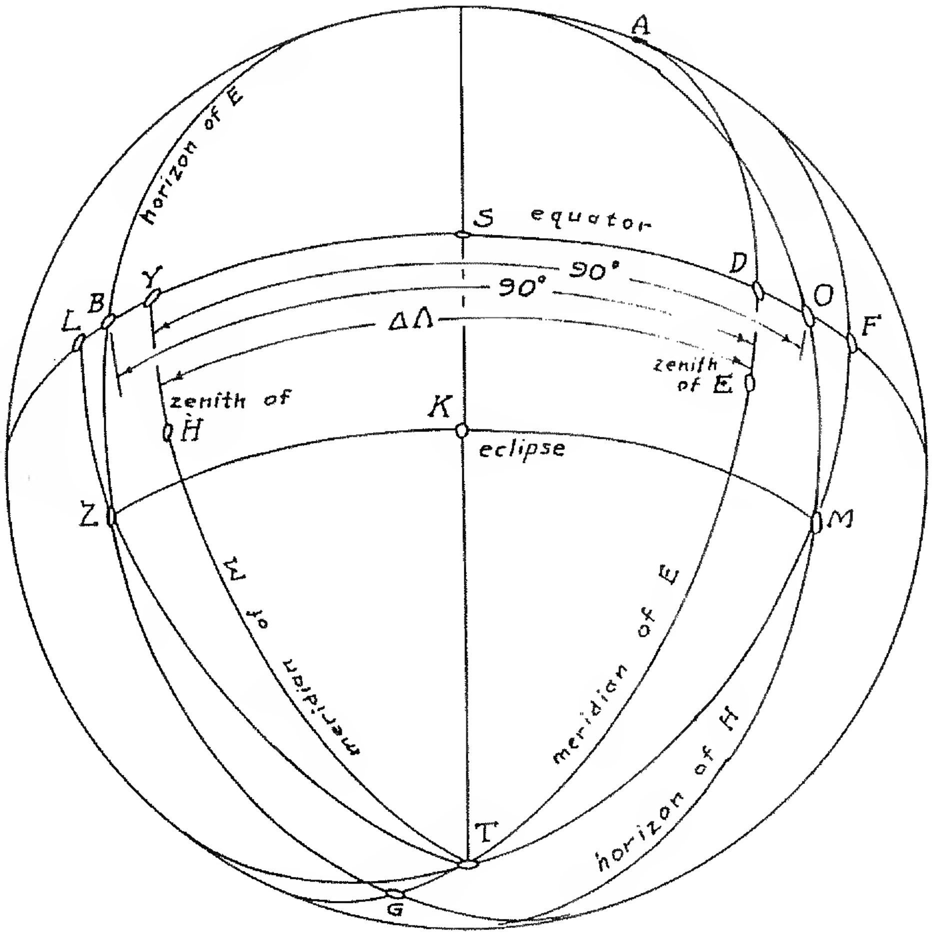


Figure C44

57. Another Digression, on the Muʿtazila (185:12 – 186:16)

In Aristotle's De Caelo (IV, 1; p. 329) is a reference to the antipodes. Perhaps it was Bīrūnī's mention of the same topic (in 185:10) which reminded him of the Muʿtazilites and their attitude toward Aristotle, and which called forth his diatribe against them. De Caelo II, 4 (p. 161) has a demonstration of the necessarily spherical shape of the element water. The statement that water "takes on a square shape in square vessels, . . ." does not seem to be a quotation, but the notion is clearly present. De Caelo III, 8 (p. 319) says, "The shape of all the simple bodies is observed to be determined by the place in which they are contained, particularly in the case of air and water".

The reasons for Abū Rayḥān's antipathy toward the Muʿtazilites can only be a matter for conjecture. Perhaps he was put off by the tendency of these scholastics of medieval Islam to engage in hair-splitting dialectic over issues which he regarded as vacuous. By his time Sunni orthodoxy tended to oppose the Muʿtazila, whereas it was still influential among the Shīʿa. Maḥmūd of Ghazna was a fierce champion of orthodoxy, and Bīrūnī's attitude may have had a political cast.

The aphorism quoted in 186:1 can also be translated as: "You have not denied it". The Muʿtazila maintained that if an opponent fails to disprove a logical proposition the proposition is valid. The quotation may refer to this doctrine, i.e. "You have not refuted an assertion, hence it stands".

The Abū Hāshim of this passage was a son of Abū ʿAlī Muḥammad b. ʿAbd al-Wahhāb al-Jubbāʾī, leader of the Baṣra school of Muʿtazilite theology. Abu Bishr Mattā b. Yūnis al-Qunnāʾī (d. 940) was a leading figure among the Christian Aristotelians of Baghdad and the first logician of his time. The Yaḥyā b. ʿAdī mentioned in 170:5 (Section 53) was a student of Abū Bishr's.

The implications of Abū Bishr's action are not clear. It may be that he invited Abu Hāshim to taste some of his saliva, it being as insipid as Abu Hāshim's speech.

(See EI, vol. 3, pp. 787–793; EIne, vol. 2, pp. 569–576, 779; GAL, GI, p. 207; Meyerhoff, pp. 415–7).

58. Various Remarks Concerning Lunar Eclipses (186:17 – 190:17)

Bīrūnī now draws together the preceding sections by the remark that if a time measurement has been made with respect to the local horizon it can be converted into a meridian determination, that is, time before or after midnight. Then, as we would put it, the algebraic difference between the two times is the longitude difference between the localities, the place of earlier occurrence being more eastern.

He next takes up an allegation (188:12) that the beginning of an eclipse cannot be observed if it occurs at sunset, nor the end at sunrise. His statement that solar parallax at the horizon is less than three minutes (of arc) and therefore negligible is consistent with Almagest V, 18, where a table gives the maximum solar parallax as $0;2,51^{\circ}$. In the same table the minimum entry for lunar parallax on the horizon is $0;53;32^{\circ}$ which is indeed more than five sixth's of a degree (189:1) and hence non-negligible.

The effect of parallax is to pull the celestial object slightly down towards, or under, the horizon, thus making it more difficult to see the eclipse. Parallax decreases as the object recedes from the earth, but on the other hand the farther the moon is from the earth the less chance does it have of entering the shadow cone. So these two effects oppose each other. With the sun, the greater its distance, the more nearly cylindrical is the shadow, and the more nearly horizontal is its top element at sunset. Hence the chance of seeing the eclipse beginning increases with solar distance. The effect of refraction is to permit the observer to see slightly down below the horizon, hence somewhat counteracting the effect of parallax. Without effort to compare these sometimes opposing and sometimes reinforcing phenomena, Bīrūnī concludes that they rarely inhibit seeing the eclipse.

The situation for the end of a sunrise eclipse is symmetrical.

Atmospheric refraction is indeed discussed by Ptolemy, in Optics V, 23–30, so the reference at 189:14 is accurate.

The eclipse phases next receive attention (190:1). If the eclipse is total there are five times involved:

(1) first contact,
(2) beginning of totality,
(3) the middle of the eclipse,
(4) end of totality, and
(5) clearance.

Since no one of these is in practice a sharply defined event, it is important to observe each at both localities and to compare cognate pairs, or for each locality to calculate the time of (3) by taking the arithmetic mean of times (1) and (5), and also the mean of (2) and (4).

These times may be measured by clepsydras, but sand clocks are regarded as preferable because of variations in the purity or the density of water, and in the effects of the air upon it (190:11 – 17). An alternative is to tell time by observations of fixed stars, and this topic is the subject of the next section.

59. Time Determination from a Star's Altitude (190:18 – 193:3)

If t is the time (in degrees of daily rotation) from the instant of the observation until meridian culmination (H in Figure C47), or from culmination until the observation, then the rule given in 191:10 is

$$(1) \qquad t = \operatorname{arc\,Vers}\left(\operatorname{Vers} d - \frac{\operatorname{Sin} h \cdot \operatorname{Vers} d}{\operatorname{Sin} h_{max}}\right),$$

where d is the arc of daylight of the observed star and h is its altitude at the time of the observation. The quantity h_{max}, altitude at culmination, does not require an additional observation if a table of fixed star declinations is at hand, since $h_{max} = \delta + \bar{\varphi}$.

The term "day-sine" (sahm al-nahār, lit. "the arrow of daylight") for Vers d entered Islamic astronomy from India. Also appearing without definition are the "triangle of the day" and the "triangle of time", being right triangles with acute angles of φ and $\bar{\varphi}$ and having $\operatorname{Sin} h_{max}$ and $\operatorname{Sin} h$ respectively as the legs adjacent to the angle φ. Bīrūnī uses them from time to time apparently regarding them as commonly known (cf. Section 104 below). To derive the rule he uses the similarity of these triangles to write

$$(191:18) \qquad OL/LF = TH/HZ, \quad \text{or} \quad \frac{OL \cdot HZ}{TH} = LF = KZ.$$

So

$$HK = HZ - KZ = HZ - \frac{OL \cdot HZ}{TH},$$

or

$$\operatorname{Vers}_{\rho} t = \operatorname{Vers}_{\rho} d - \frac{\operatorname{Sin}_R h \cdot \operatorname{Vers}_{\rho} d}{\operatorname{Sin}_R h_{max}}.$$

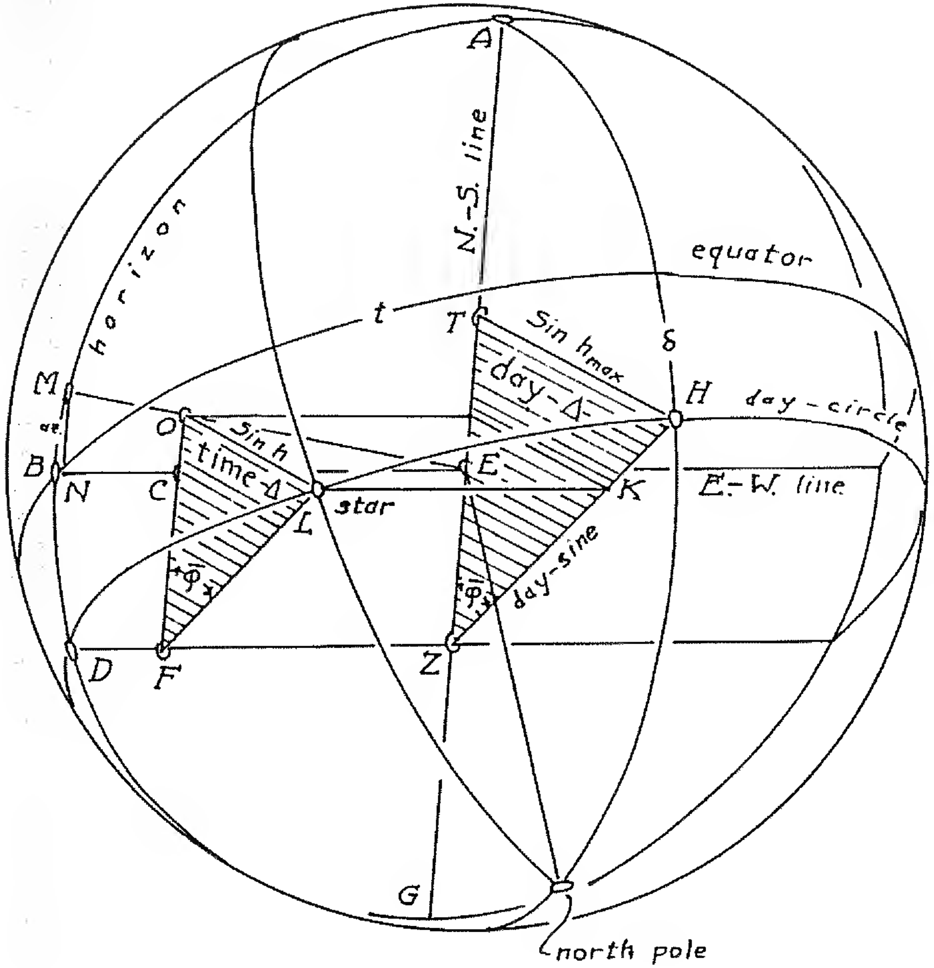


Figure C47

Note, although there is no mention of it in the text, the two different parameters associated with the trigonometric functions. They are ρ (= Cos δ), the radius of the day-circle, and the customary R, radius of the celestial sphere. But the two sines with parameter R appear as a ratio; hence their quotient can be regarded as a dimensionless number. The elimination of R leaves only functions based on ρ, so the expression (1) above is valid where the implicit parameter may be any desired constant. As usual the use of negative numbers will take care of all cases.

Formulas equivalent to (1) appear in Indian and early Islamic astronomy (see Davidian).

60. Time Determination from a Star's Azimuth (193:4 – 194:20)

Here we combine the enunciation of the very complicated rule with its derivation. In Figure C48, by application of the sine law to the right triangle GMS,

$$\text{(194:1)} \qquad \frac{\text{Sin }(SG = \bar{\varphi})}{\text{Sin } GM} = \frac{(\text{Sin } GMS) = R}{\text{Sin }(MSG = \text{az.} = ZA)},$$

whence

$$\text{Cos } \varphi \text{ Cos az.} = R \cdot \text{Sin } GM\,(= r_1,\ 193:6).$$

By the Rule of Four applied to the right triangles SHB and HML,

$$\text{(194:6)} \qquad \frac{\text{Sin } SH}{\text{Sin } \varphi} = \frac{(\text{Sin } HM) = R}{\text{Sin }(ML = \overline{GM})}.$$

So

$$\text{Sin } ML = R \cdot \text{Sin } \varphi \,/\, \text{Sin } SH\ (= r_2 = \text{Cos (arc Sin }(r_1/R)),$$

193:7).

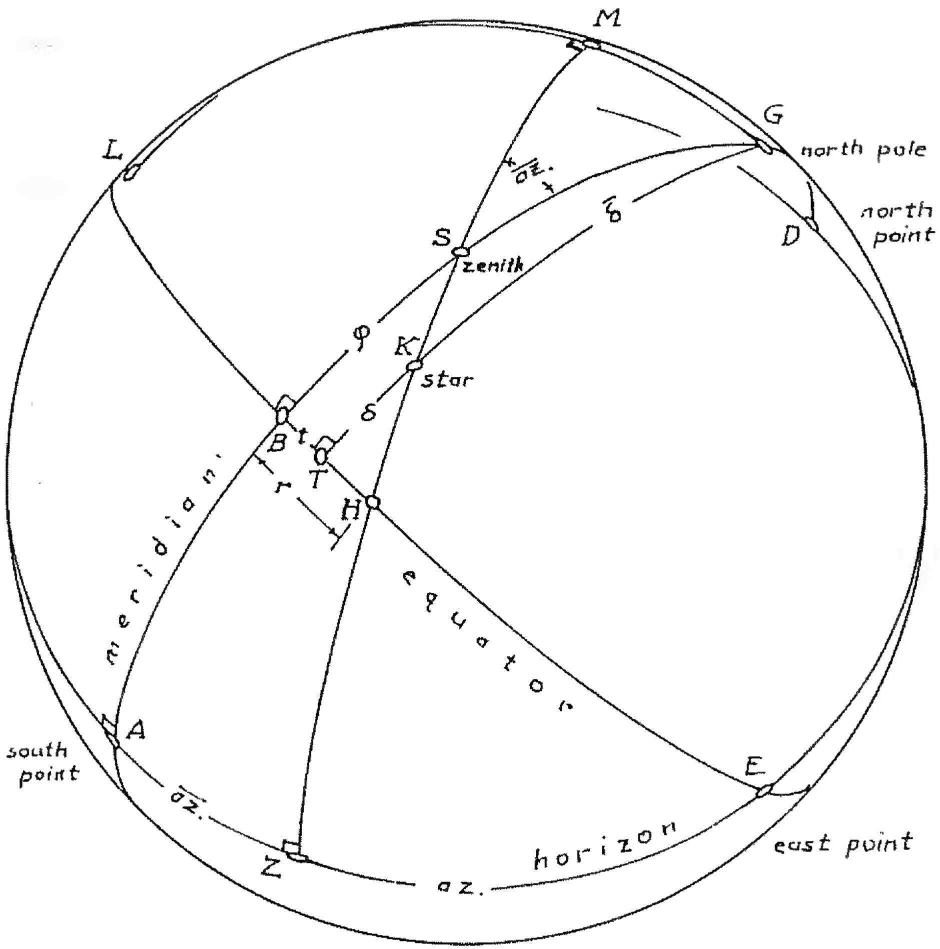


Figure C48

Also (194:9), by applying the Rule of Four to the triangles SHB and SZA,

$$\frac{\text{Sin SH}}{\text{Sin HB}} = \frac{(\text{Sin SZ}) = R}{\text{Sin (ZA} = \overline{\text{az.}})}$$

so

$$\text{HB} = \text{arc Sin}\left(\text{Sin SH} \cdot \frac{\text{Cos az.}}{R}\right) = \text{arc Sin}\left(\frac{R . \text{Sin}\,\varphi}{r_2} \cdot \frac{\text{Cos az.}}{R}\right) = r, \text{ say.}$$

(In 193:8 Bīrūnī defines r as arc $\text{Sin}\left(\frac{r_2 \cdot \text{Sin}\,\varphi}{R} \cdot \frac{\text{Cos az.}}{R}\right)$, but this is a slip.)

By the law of sines applied to the right triangle KGM

$$(194:11) \qquad \frac{\text{Sin (KG} = \bar{\delta})}{\text{Sin GM}} = \frac{(\text{Sin GMK}) = R}{\text{Sin GKM}} .$$

So $\quad R \cdot \text{Sin GM} = \text{Cos}\,\delta \cdot \text{Sin GKM} = r_1$,

and (194:14) $\quad \text{Sin GKM} = r_1 / \text{Cos}\,\delta$.

Applying the sine law to right triangle THK,

$$(194:14) \qquad \frac{\text{Sin (GKM} = \text{TKH)}}{(\text{Sin (THK} = \text{ML})) = r_2} = \frac{\text{Sin TH}}{\text{Sin (KT} = \delta)} .$$

$$(193:9) \quad \text{TH} = \text{arc Sin}\left(\frac{\text{Sin GKM} \cdot \text{Sin}\,\delta}{r_2}\right) = \text{arc Sin}\left(\frac{r_1}{\text{Cos}\,\delta} \cdot \frac{\text{Sin}\,\delta}{r_2}\right).$$

Finally

$$(194:16) \qquad t = \text{BT} = \text{BH} - \text{TH}.$$

Our Figure C48 shows the situation in the first part of the text and translation Figure 48, with $\delta > 0$ and H above the horizon. For the second drawing of Figure 48, $\delta > 0$, but H falls below the horizon. For the third drawing $\delta < 0$, and for the fourth $\delta = 0$. With all cases the same rule and demonstration hold with appropriate changes of sign.

61. Time Determination from a Star's Altitude and Azimuth (195:1 – 196:6)

When both the altitude and azimuth are at hand proceed as follows: from the similar triangles EOC and EMN in Figure C47

$$\frac{EO\ (= \text{Cos}\ h)}{OC\ (= \text{az.''share''})} = \frac{EM\ (= R)}{MN\ (= \text{Sin az.})},$$

or

$$OC = \text{Cos}\ h \cdot \text{Sin az.}/R$$

(195:7)

$$EC = (\overline{EO}^2 + \overline{OC}^2)^{\frac{1}{2}} = (\text{Cos}^2 h - \text{Sin}^2 (\text{az.}))^{\frac{1}{2}}$$
$$= KL = \text{Sin}_\rho\ t$$

where $\rho = \text{Cos}_R\ \delta$.

so

(196:6) $$t = \text{arc Sin}_R\left(\frac{R \cdot \text{Sin}_\rho\ t}{\text{Cos}_R\ \delta}\right)$$

62. Solar Longitude at the Time of the Eclipse (196:7 – 196:12)

Assume that, by whatever means, t, the time in degrees from the eclipse until the culmination of the particular fixed star, has been calculated. Subtraction of t from the right ascension of the star gives the right ascension of upper midheaven at the time of the eclipse, say.

Assume also as known the solar longitude at sunset and the right ascension of upper midheaven at sunset, say $\alpha\ (t_e)$. (It would be easier to determine these two values as of the preceding noon or midnight, but the subsequent procedure would not be changed.)

Then $$\Delta\lambda = \left[\alpha\,(t_e) - \alpha\,(t_o)\right] \frac{\dot{\lambda}}{360^\circ} \qquad (196:9)$$

will be the proper motion of the sun from sunset until the time of the eclipse, where $\dot{\lambda}$ is the buht, the solar motion in longitude in degrees of arc per day. Add $\Delta\lambda$ to the solar longitude at sunset to obtain its longitude at the time of the eclipse. Subtract 180° from this to obtain the position of the eclipse on the ecliptic.

63. Spherical-Astronomical Nomenclature (196:13 – 198:20)

Of the documents mentioned in this passage, the zīj of al-Khwārizmī survives only in a Latin translation of an Arabic recension (see Khwārizmī Zīj, also Section 20 above; for the Sindhind, Section 29). The form translated in 196:17 as "equator" is khaṭṭ al-istiwā', usually signifying the "terrestrial equator".

There are two extant versions of zījes by Ḥabash (Section 37), neither of which has been published. The "star's course" is majrā al-kawkab, which can also be translated as the "day-circle" of a star.

The zīj of al-Nairīzī (Section 22) is not extant, but it was based on that of al-Battānī, which has been published (see Battānī, also Section 22).

In medieval Islamic astronomy the term mayl usually denoted the declination of a point on the ecliptic. This is a more limited meaning than that conveyed by the word "declination" as used nowadays. The declination of an ecliptic point can be found from a declination table such as that given in Almagest I, 15. When the use of the word mayl was extended to mean the distance to the celestial equator of an object not on the ecliptic (declination in the modern sense), special definition was required. This Bīrūnī proceeds to give, together with techniques for calculating it.

In Figure C49 O is a pole of HZKMD and J is a pole of OSMG. Hence GM = SO and EO = ZB (198:1). We assume as given the ecliptic coordinates β and λ (= 180° – EZ) of the star K. By inverse use of a table of right ascensions HE may be found from the known EZ (197:12). That is, enter the column of the dependent variable (α) with λ . Then the corresponding value of the independent variable will be 180° – HE. Now enter a declination table (like that of the Almagest referred to above) with EH as argument; the corresponding "declination" will be HZ (197:14). Add it (algebraically) to the given β (= KZ) to obtain HK. In like manner, enter the declination table with OE = ZB = $\overline{EZ}$ as argument to obtain OS = δ(OE). Hence find MS = $\overline{OS}$ = $\overline{GM}$.

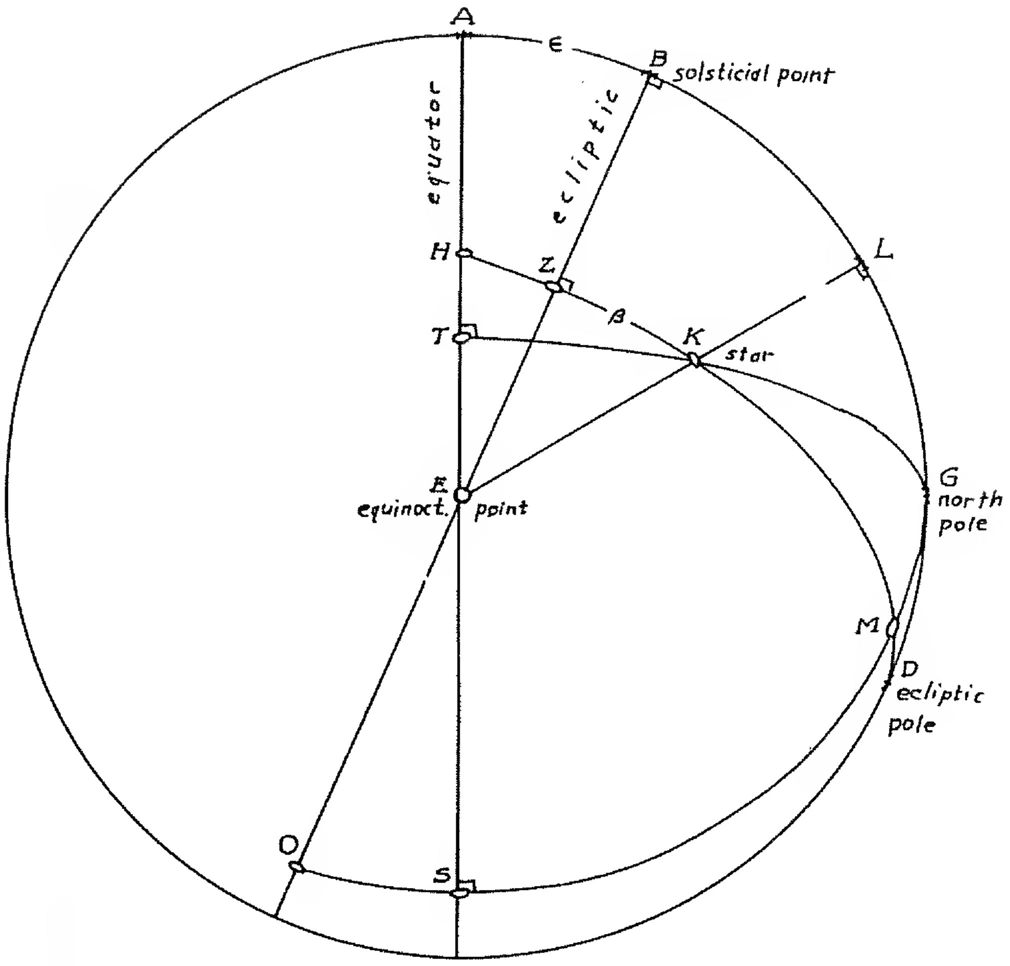


Figure C49

By application of the Rule of Four to triangles HKT and HMS

$$\frac{\text{Sin HK}}{\text{Sin (KT} = \delta)} = \frac{(\text{Sin HM}) = R}{\text{Sin MS}}$$

From this, HK and MS having been calculated, KT can be found (198:5). This is the star's declination in the modern sense. The process of finding it is given in general terms in the rule of 196:19 – 197:5.

An alternative procedure is to apply the Rule of Four to the triangles DKL and DZB to obtain

$$(198:6) \qquad \frac{\text{Sin (DK} = \bar{\beta})}{\text{Sin KL}} = \frac{(\text{Sin DZ}) = R}{\text{Sin (ZB} = \lambda - 90^{\circ})},$$

or

$$\text{Sin KL} = \text{Cos KE} = \frac{\text{Cos } \beta \cdot \text{Sin ZB}}{R}.$$

By the Rule of Four applied to triangles KEZ and LEB,

$$(198:11) \qquad \frac{(\text{Sin KE}) = \text{the "part"}}{\text{Sin (KZ} = \beta)} = \frac{(\text{Sin LE}) = R}{\text{Sin LB}}.$$

So

$$\text{Sin LB} = \frac{R \cdot \text{Sin } \beta}{\text{Sin KE}}.$$

By the Rule of Four applied to triangles LAE and KTE,

$$(198:17) \qquad \frac{\text{Sin (LA} = \text{LB} + \varepsilon)}{(\text{Sin EL}) = R} = \frac{\text{Sin KT}}{(\text{Sin EK}) = \text{the "part"}},$$

or

$$\delta_K = \text{KT} = \text{arc Sin}\left(\frac{\text{Sin LA} \cdot \text{Sin EK}}{R}\right).$$

64. Ecliptic Degree of Culmination (199:1 – 200:11)

In order to locate O (Figure C50), the ecliptic point which transits across the meridian simultaneously with the star S, one may apply the Rule of Four to the triangles FHY and FTL to obtain

$$(199{:}4) \qquad \frac{\text{Sin }(FH = \overline{HT})}{\text{Sin }(HY = \overline{HK})} = \frac{(\text{Sin } FT) = R}{\text{Sin }(T = \overline{TK} = \delta_K)} ,$$

or

$$\text{Cos } HT = \frac{R \cdot \text{Cos } HK}{\text{Cos } \delta_K}$$

(Figures C49 and C50 show the same configuration except that in the latter the great circle FYL having the star as pole has been added, and O is assigned to different points). In the last expression HK was just calculated in Section 63, as was δ_1. So HT, the "equation" can be found.

Alternatively, apply the Rule of Four to the triangles KHT and KGM to obtain

$$(199{:}8) \qquad \frac{\text{Sin } KH}{\text{Sin } HT} = \frac{\text{Sin }(KG = \overline{KT} = \bar{\delta}_K)}{\text{Sin } GM} .$$

As remarked before, KH and δ_K were obtained above, as was GM (or its complement). Again the only unknown left in the expression is HT, and it can be calculated.

But the point H likewise was located in the preceding section. Apply HT to it algebraically in order to locate T. Then the inverse right ascension of T is O, the desired ecliptic point of transit.

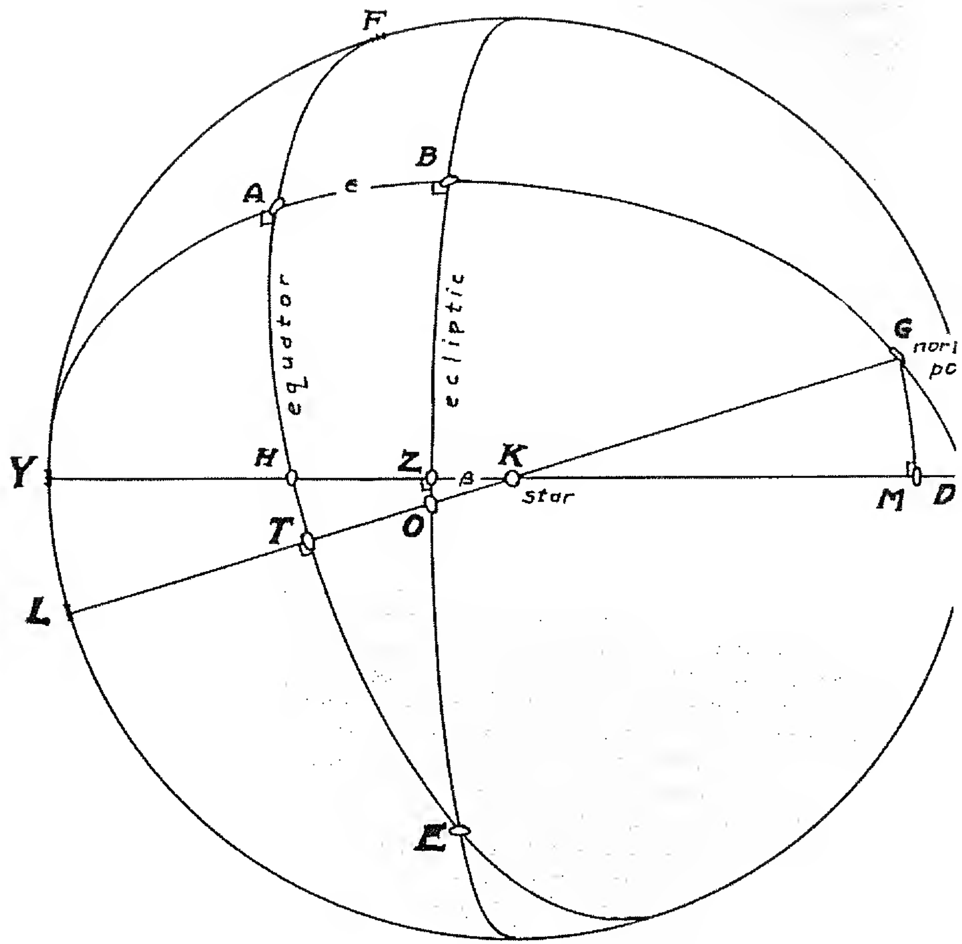


Figure C50

65. Avicenna Determines the Longitude of Gurgān (201:1 – 203:9)

Shams al-Maʿālī (Sun of the Heights) was the honorific title bestowed upon Qābūs b. Washmagīr (d. 1012) the Ziyārid ruler of Gurgān. He maintained the young Avicenna (b. 980) at his court after the philosopher's departure from his native Bukhara. Qābūs was also for a time Bīrūnī's patron, who dedicated to him the Chron. The name of Qābūs' daughter, Zarrayn Kīs, is Persian for "Golden Purse". Avicenna's lost treatise is not a unique example of a medieval scientific book being dedicated to a Muslim noblewoman. Bīrūnī's Tafhīm was addressed to the Khwārazmian lady Rayḥāna, bint al-Ḥasan.

In order to appreciate Bīrūnī's comments it is useful to state the operations which it would have been necessary for Avicenna to carry out in order to obtain the results he claimed.

The observational part of the job would be straightforward – on a suitable night the meridian altitude of the moon was recorded – although it would not have been trivial to obtain a reading reliable within a minute of arc. But then the real work commences. Presumably Avicenna had a zīj calculated with Baghdad as a base meridian. Assuming a $\Delta\Lambda$ of 8° he could have calculated for, say time t_1, midnight at Gurgān, λ_m, β_m, and the moon's hour angle, its right-ascensional distance from the meridian of Gurgān. The coordinates must be converted from true to apparent, that is, the effect of parallax must be considered. In general, the moon will not turn out to have been on the meridian at midnight. If it had passed the meridian at that time, estimate the time since it crossed, and subtract this from t_1 to obtain t_2. If, on the other hand, the moon had not reached the meridian at midnight, obtain t_2 by adding to t_1 the estimated additional time between it and transit. Now repeat the entire operation with t_2 replacing t_1 and see how close the moon is to the Gurgān meridian. If it is still substantially off choose a t_3 and continue. When the iterative process has converged satisfactorily, calculate, by the methods of Section 63 or otherwise, the moon's declination, and add it to the $\bar{\varphi}$ of Gurgān. It was presumably the end result of such a procedure which yielded the $h = \bar{\varphi} + \delta_m = 80;4^{\circ}$ of 201:9.

Now note whether the lunar declination was increasing or decreasing at the time of transit, and estimate its rate of change. If it was increasing, increase the final t_n by a Δt sufficient to cause an increment of $0;2^{\circ}$ in δ_m. If it was decreasing, Δt will be negative. Avicenna's Δt was $1;20^{\circ}$ of daily rotation.

The modern value for the longitudes of Gurgān and Baghdad are $54;29^{\circ}$ and $44;29^{\circ}$ respectively. Hence the actual $\Delta\Lambda$ is $10;3^{\circ}$, and

Avicenna's $9;20^{\circ}$ is better than the traditional $8;0^{\circ}$. But the improvement was probably fortuitous.

The suggestion attributed to Ḥabash (Section 37) in 202:8–17 is subject to many of the objections adduced against Avicenna. However a lunar eclipse is probably a more sensitive phenomenon than lunar declination at transit.

66. Al-Hāshimī Determines the Longitude of Raqqa (203:10 – 204:12)

This passage adds a bit to what little information is available concerning the scientist Muḥammad b. ʿAbd al-ʿAzīz al-Hāshimī, namely that he observed the lunar eclipse of 16 November 932 (Oppolzer No. 3307) at Raqqa, a great and ancient city of upper Mesopotamia, on the bank of the Euphrates (Le Str., p. 101). Al-Hāshimī's al-Zīj al-Kāmil is not extant, but Bīrūnī apparently had a copy of it (Survey, p. 135; Suter, p. 79). Al-Hāshimī also wrote a commentary on the zīj of al-Khwārizmī (Fazārī, p. 119).

Since $0;28 \times 15^{\circ/h} = 7;0^{\circ}$, the latter is al-Hāshimī's $\Delta\Lambda$ for Baghdad-Raqqa, not $7;5^{\circ}$ as in the text and translations. The misprint is due to a confusion of the Arabic sexagesimal zero symbol with the letter-numeral ha' = 5, which it resembles (see Irani).

The modern coordinates of the two places, together with two versions of their latitudes according to al-Battānī (vol. II, pp. 41, 42), are

	Longitude	Latitude modern	Latitude al-Battānī (Taḥdīd)	Latitude al-Battānī (zīj)
Baghdad	$44;26^{\circ}$	$33;20^{\circ}$	$33;25^{\circ}$	$33;9^{\circ}$
Raqqa	$39;3^{\circ}$	$35;57^{\circ}$	$36;1^{\circ}$	$36;0^{\circ}$

So Bīrūnī is correct in stating that Raqqa is west of Baghdad; the actual $\Delta\Lambda$ is $5;23^{\circ}$. If, as he says, both times were taken from the respective local sunsets, the case is that discussed at 180:1 (Section 56), and the equation of daylight at both places should enter.

As for Alexandria, the reported time difference is $0;50^h = 0;50^h \times 15^{\circ/h} = 12;30^{\circ}$. The modern coordinates of Alexandria are

Longitude: $29;55^{\circ}$ Latitude: $31;13^{\circ}$.

Hence the correct $\Delta\Lambda$ between Alexandria and Raqqa is 9;8°. The latitude Bīrūnī reports for Alexandria, 30;58°, is quite inaccurate. This value is confirmed as Bīrūnī's by Abū al-Fidā' (p. 155), and the entry in the Canon (p. 555) should be restored to it. Also it is quite close to Ptolemy's (in the Geogr.) 31;0°. But Bīrūnī is right in stating that the equations of daylight should be considered if the times were with respect to the local horizons.

67. A Rule from al-Sarakhsī's Zīj (204:13 – 206:7)

Muḥammad b. Isḥāq and his zīj are known only through various remarks in Bīrūnī's writings (cf. Survey, p. 131). From these it is clear that the zīj depended primarily on the Sindhind tradition, and this additional mention strengthens the case. The term qubbat Arīn (cupola of Arīn) or qubbat al-arḍ (cupola of the earth), here shortened to qubba, is the Muslim designation for Ujjain, the Indian Greenwich. Ujjain was transliterated into Uzain in Arabic, whence a dot dropped from over the letter za' converted it into a ra', hence Arīn. The vowel sounds are not normally indicated in the script. Al-Sarakhsī's longitudes were reckoned from the Cupola, by his time taken to be on the equator, although, of course, Ujjain is not. It was assumed that longitudes of inhabited localities reached a quadrant on either side of the Cupola (Battānī, vol. 2, p. 349; Tafhīm, ed of Wright, p. 140). In the Canon (p. 547), however, Bīrūnī gives the Cupola a longitude of 100;50° on the equator and says it is the island of Lañkā.

To explain al-Sarakhsī's rule turn to Figure C51 where D_1, the zenith of the Cupola, is seen to be on the equator. The situation is essentially that discussed in Section 55. In Figure C51 call

t_D = BS, the time from sunset at the Cupola until the eclipse,

and

t_H = FS, the time from sunset at H until the eclipse.

Then for an arc of half daylight exceeding a quadrant (204:16) i.e. $\delta_F > 0$, the rule should be

$$\Delta\Lambda = t_H - FO - t_D,$$

where FO is the equation of half daylight. Al-Sarakhsī has said add where he should have said subtract.

When $\delta < 0$, as indicated by primed letters on the figure, the sign of the daylight equation will be reversed.

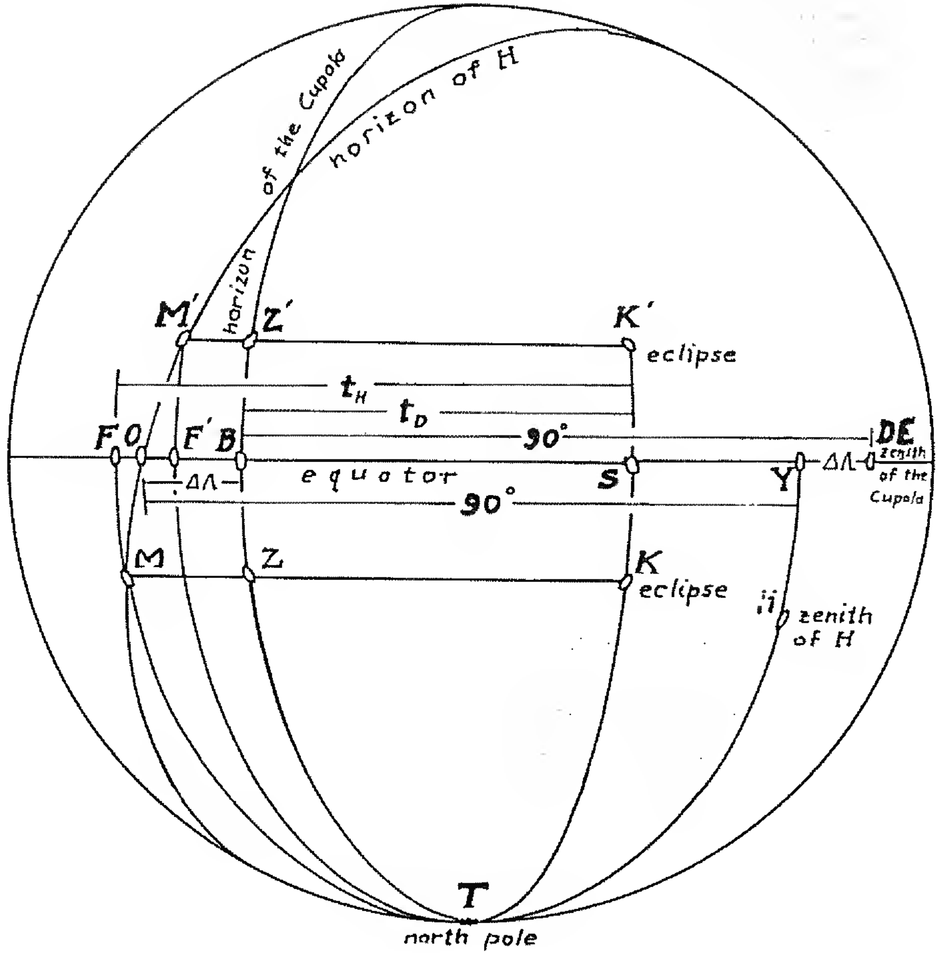


Figure C51

68. Distance and Azimuth from the Coordinates of Two Localities (206:8 – 215:6)

This section gives solutions for a problem basic to the entire treatise: given the latitudes of two localities (E and H in Figure C52) and the difference between their longitudes ($\Delta\Lambda$) calculate the great circle distance (EH) between them and the direction of one from the other (say LB = az).

By the Rule of Four applied to the triangles HTK and TYD,

$$(207:10) \qquad \frac{\sin\ (HT = \bar{\varphi}_H)}{\sin\ (HK = eq.\ \Lambda)} = \frac{(\sin\ TY) = R}{\sin\ (YD = \Delta\Lambda)}$$

where HK is called the "equated" or "modified" (mu'addal) longitude. So

$$\sin\ (eq.\ \Lambda) = (\cos\ \varphi_H \cdot \sin\ \Delta\Lambda)\ /\ R$$

By the Rule of Four applied to the triangles BHY and BKD,

$$(207:14) \qquad \frac{\sin\ (BH = \overline{eq.\ \Lambda})}{\sin\ (HY = \varphi_H)} = \frac{(\sin\ BK) = R}{\sin\ (KD = eq.\ \varphi)}.$$

Therefore

$$\sin\ (eq.\ \varphi) = R \cdot \sin\ \varphi\ /\ \cos\ (eq.\ \Lambda).$$

Application of the Rule of Four to triangles BHL and BKA gives

$$(208:2) \qquad \frac{\sin\ (BH = eq.\ \Lambda)}{\sin\ (HL = HE)} = \frac{(\sin\ BK) = R}{\sin\ (KA = \varphi_E - eq.\ \varphi)}.$$

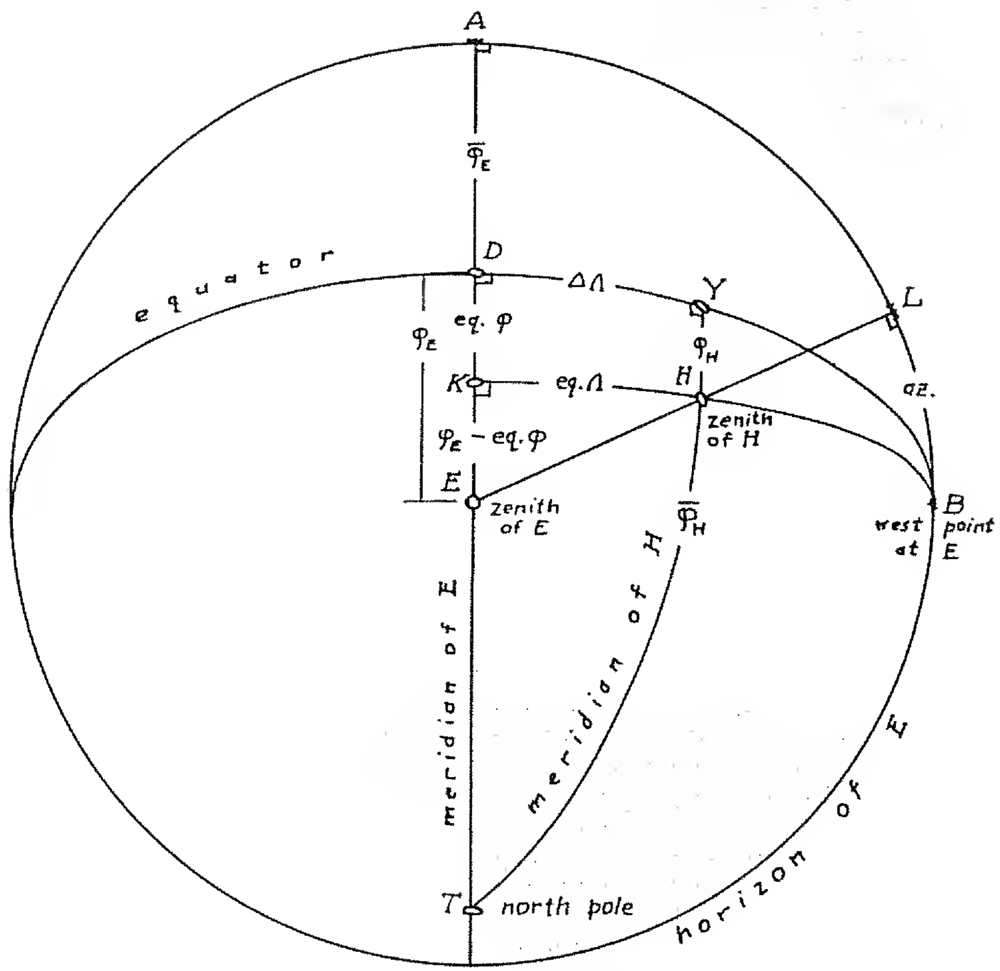


Figure C52

Hence the distance between the two localities is

$$HE = \text{arc Cos}\left[\text{Cos}\,(\text{eq}.\Lambda)\ \text{Cos}(Q - \text{eq}.\varphi)/R\right].$$

Now, applying the Rule of Four to triangles HER and ELA,

$$(208:7)\qquad \frac{\text{Sin HE}}{\text{Sin}\,(HK = \text{eq}.\ \Lambda)} = \frac{(\text{Sin EL}) = R}{\text{Sin}\,(LA = \overline{az}.)}$$

So the direction of H from E is

$$az. = \text{arc Cos}\,(R \cdot \text{Sin}\,(\text{eq}.\ \Lambda)\,/\,\text{Sin HE}).$$

Alternatively, using the Rule of Four with triangles HBL and HEK,

$$(208:12)\qquad \frac{\text{Sin}\,(HB = \text{eq}.\ \Lambda)}{\text{Sin}\,(BL = az.)} = \frac{\text{Sin HE}}{\text{Sin}\,(EK = \varphi_E - \text{eq}.\cdot\varphi_E)},$$

whence

$$az. = \text{arc Sin}\left[\text{Cos}\,(\text{eq}.\ \Lambda)\cdot \text{Sin}\,(\varphi_E - \text{eq}.\varphi)/\,\text{Sin HE}\right]$$

Bīrūnī now remarks (209:7) that if locality H is the city of Mecca, then the problem just solved becomes that of the qibla, the direction of the Muslim prayers. He reverts to this at the end of the book, meanwhile discussing the coordinates of Mecca. Its latitude is commonly taken as 21°, he says, although it is really a fraction more. Manṣūr b. Ṭalḥa found it to be 21;40°, as did the Caliph Ma'mūn as reported by Ḥabash. Others make it 21;20°. Since the accurate value is 21;26°, these last had the best of it.

As for its longitude, the $\Delta\Lambda$ between it and Baghdad was found by Ma'mūn (as reported by Ḥabash, see Section 20) and by Ibn Talḥa to be 3°. The accurate amount is 4;37°.

Concerning Manṣūr b. Ṭalḥa, see Section 23; for Ma'mūn, Section 20. The book of Ḥabash (Section 37) referred to here, "On Distances and (Celestial) Bodies" is not extant. It was doubtless an example of a class of ancient and medieval studies giving the size of the earth, and the size and distances of the heavenly bodies. A number of such works exist.

In connection with the use of Jerusalem as the Jewish (and temporary Muslim) qibla, it is of interest to recall the Western European notion that Mount Purgatory was located at the antipodal point to Jerusalem (Dante, pp. 85, 350).

The remark (at 210:11) about neglecting what must be done five times a day is a reference to the five daily prayers incumbent upon the Muslim, for which he needs the qibla.

69. Length of a Degree Along a Meridian (211:2 – 215:6)

The notion that a degree along the meridian is $66\frac{2}{3}$ miles (211:13) is ancient, reaching back at least to the fourth century A.D. It stems from an erroneous conversion factor of 7.5 between miles and stadia. Ptolemy's Geography takes the circumference of the earth as 180,000 stadia. Use of the above factor makes this 180,000/7.5 = 24,000 miles, and this number was widely attributed to Ptolemy by Muslim geographers. Hence the length of a degree is 24,000 / 360° = $66\frac{2}{3}$ miles.

The curious and apochryphal story of a survey by which this result was obtained seems to have come from a garbled report of one of the surveys ordered by the caliph al-Ma'mūn (Section 20) and described later in the Taḥdīd. Ibn Yunus (Comm. Vol., p. 10) quotes Sanad b. ʿAlī (Section 20) as stating that he and Khālid (Section 10) operated between Raqqa (Section 22) and Palmyra, while ʿAlī b. ʿĪsā and al-Baḥtarī worked elsewhere, presumably near Sinjār. But both parties obtained a degree's length of 57 miles, not the "Ptolemaic" value. By the time the story got to al-Makkī (211:19, see Section 23) and ʿAlī b. Sahl Rabbān al-Ṭabarī (fl. 835, Comm. Vol., p. 6; Suter, p. 14) only the Raqqa-Palmyra locale remained, the investigators' names and data having disappeared. Someone then cooked up a derivation of the Ptolemaic 66;40 along the following lines:

Coordinates of the two cities are

	Longitude			Latitude		
	Ptol.	Geogr.	modern	Ptol. Geogr.	Bīrūnī	Modern
Tadmor (Palmyra)	71;40°	(V, 15, 24)	38;15°	34;0°		34;36°
Raqqa (Nikephorion)	73;12°	(V, 18, 6)	39;3°	35;20°	36;1°	35;57°
Difference				1;20°		1;21°

Note that the latitudes given by al-Makkī and al-Ṭabarī must stem from Ptolemy. However, an essential part of the story is that the two towns have the same longitude, and this is denied both by Ptolemy's Geography and by the facts. Assuming them to be on the same meridian the length of a degree is the

$$\text{distance}\ /\ \Delta\varphi = 90 \text{ miles}\ /\ 1;20^\circ = 67\tfrac{1}{2} \text{ miles},$$

which is too much. As Bīrūnī remarks what is needed is

$$90 \text{ miles}\ /\ 1;21^\circ = 66\tfrac{2}{3} \text{ miles}.$$

The fact that the actual $\Delta\varphi$ is 1;21° is purely fortuitous. (See Scritti, pp. 293 and 412-6).

Concerning al-Fazārī and his zīj (211:21) see Section 48 above. His Indian value for the circumference of the earth

$$
\begin{aligned}
& 6,600 \text{ farsakhs} \quad (=3,300 \text{ yojanas}) \\
= \; & 6,600 \times 16,00 \text{ cubits} \\
= \; & 360^\circ \times 55 \text{ miles }/^\circ \quad = \quad 19,800 \text{ miles}
\end{aligned}
$$

is found in Lalla's Siṣyadhīvṛddhidatantra I, 1,56. It may be rounded off from $6,597\frac{9}{25} = 2 \times 3,298\frac{15}{25}$ which occurs later in the Taḥdīd (228:13) and which stems from the Āryabhaṭīya (see Fazārī, p. 121).

His value attributed to Hermes,

$$
\begin{aligned}
& 9,000 \text{ farsakhs} \\
= \; & 9,000 \times 12,000 \text{ cubits} \\
= \; & 360^\circ \times 75 \text{ miles }/_\circ
\end{aligned}
$$

has been found in no other source.

The strange figure of Hermes Trismegistos, the Hellenistic name of the Egyptian god Thoth, entered the Islamic world as a hero (or three heroes) of ancient times, author of books on philosophy, science, and magic, and preserver of ancient knowledge from the Flood (EIne, Vol. 3, pp. 463-5).

The circumference of

$$
\begin{aligned}
& 12,000 \text{ farsakhs} \\
= \; & 12,000 \times 3 \text{ miles} \\
= \; & 360^\circ \times 100 \text{ miles }/_\circ
\end{aligned}
$$

likewise we find nowhere else.

For remarks on al-Hirawī and his book (212:11) see Sections 23 and 52. He also has somehow garbled the report of the meridian measurements made by al-Ma'mūn, for the result he gives, 56;40 miles /o is indeed one of the numbers associated with the survey, but the distance between the two towns could not have been employed as he says, as Bīrūnī points out. Their coordinates are

	Modern Long.	Modern Lat.	Bīrūnī Lat.
Sāmarrā	43;52°	34;13°	34;12°
Baghdad	44;26°	33;20°	33;20° or 25
difference	0;34°	0;53°	0;52° or 0;57°

and neither premise was valid: the one locality is not due north from the other, nor is their difference in latitude a degree.

The "black cubit" (212:61). a unit set by al-Ma'mūn himself, measured some 54.04 cm. (Hinz, pp. 55, 61).

The parameter said to be "Greek" in 213:13 is found in Ptolemy's Geography. Taking the value for the circumference of the earth given at the beginning of the section,

$$180,000 \text{ stadia } / \, 360^\circ \quad = \quad 500 \text{ stadia } /^\circ .$$

The uncertainty about the actual length of this unit persists to this day.

Concerning Ḥabash and his book on distances, see Sections 37 and 68.

Mosul (213:17, properly Mawṣil) is on the Tigris well above Sāmarrā, and across the river from the ancient site of Nineveh. The modern town of Sinjār is about seventy miles west of Mosul. To the north are mountains; doubtless the survey was made in the flat country to the south.

ʿAlī b. ʿĪsā, as his nickname indicates, was best known as a maker of astronomical instruments. His treatise on the astrolabe is extant in several copies (Suter, p. 13).

Aḥmad b. al-Buḥtari (214:2) is otherwise unknown. Ibn Yunus (in, e.g. Suter, p. 209) reports an ʿAlī b. al-Buḥtarī, probably the same individual, as being engaged in the survey at Sinjār.

Yaḥyā b. Aktham (d. 857) was a well-known jurist, the chief justice of Baṣra (Suter, p. 30).

Concerning al-Ṣaghānī, see Section 24; for Thābit b. Qurra, Section 11.

Aḥmad b. Muḥammad b. Kathīr (or Kuthayr) al-Farghānī, of Central Asiatic origin, was an Abbasid astronomer. During the reign of al-Mutawakkil (847-61) he was sent to Egypt to oversee the construction of a nilometer (Suter, p. 18).

From the Ma'mūnic surveys there emerged two results for the length of a degree, 56 and 56;40 miles of four thousand cubits each, both attested in the literature. On pp. 216-217 Bīrūnī gives a sexagesimal conversion table for carrying miles (or farsakhs, each equal to three miles) into degrees. The column headed Ḥabash is the 56 unit; that headed al-Farghānī is the 56;40. Bīrūnī himself uses the latter value in the computations later on.

Displayed below are the results of recomputing the table with the IBM 1620 computer at the American University of Beirut. Here the calculation has been carried one sexagesimal digit beyond the results of the text. In the latter all the entries under Ḥabash are correct and have been properly rounded. However, in the column under

E 2: SEXAGESIMAL CONVERSION TABLE FROM FARSAKHS AND MILES TO DEGREES ALONG THE MERIDIAN

Farsakhs	Miles	Ḥabash	Al-Farghānī
	1	0; 1, 4, 17, 8	0; 1, 3, 31, 45
	2	0; 2, 8, 34, 17	0; 2, 7, 3, 31
1	3	0; 3, 12, 51, 25	0; 3, 10, 35, 17
	4	0;4, 17, 8, 34	0; 4, 14, 7, 3
	5	0; 5, 21, 25, 42	0; 5, 17, 38, 49
2	6	0; 6, 25, 42, 51	0; 6, 21, 10, 35
	7	0; 7, 30, 0, 0	0; 7, 24, 42, 21
	8	0; 8, 34, 17, 8	0; 8, 28, 14, 7
3	9	0; 9, 38, 34, 17	0; 9, 31, 45, 52
	10	0;10, 42, 51, 25	0;10, 35, 17, 38
	11	0;11, 47, 8, 34	0;11, 38, 49, 24
4	12	0;12, 51, 25, 43	0;12, 42, 21, 10
	13	0;13, 55, 42, 51	0;13, 45, 52, 56
	14	0;15, 0, 0, 0	0;14, 49, 24, 42
5	15	0;16, 4, 17, 8	0;15, 52, 56, 28
	16	0;17, 8, 34, 17	0;16, 56, 28, 14
	17	0;18, 12, 51, 25	0;18, 0, 0, 0
6	18	0;19, 17, 8, 34	0;19, 3, 31, 45
	19	0;20, 21, 25, 43	0;20, 7, 3, 31 *
	20	0;21, 25, 42, 51	0;21, 10, 35, 17
7	21	0;22, 30, 0, 0	0;22, 14, 7, 3**
	22	0;23, 34, 17, 8	0;23, 17, 38, 49
	23	0;24, 38, 34, 17	0;24, 21, 10, 35
8	24	0;25, 42, 51, 26	0;25, 24, 42, 21
	25	0;26, 47, 8, 34	0;26, 28, 14, 7
	26	0;27, 51, 25, 43	0;27, 31, 45, 52
9	27	0;28, 55, 42, 51	0;28, 35, 17, 38*
	28	0;30, 0, 0, 0	0;29, 38, 49, 24
	29	0;31, 4, 17, 8	0;30, 42, 21, 10
10	30	0;32, 8, 34, 17	0;31, 45, 52, 56

Farsakhs	Miles	Ḥabash	Al-Farghānī
	31	0;33, 12, 51, 26	0;32, 49, 24, 42 *
	32	0;34, 17, 8, 34	0;33, 52, 56, 28
11	33	0;35, 21, 25, 43	0;34, 56, 28, 14
	34	0;36, 25, 42, 51	0;36, 0, 0, 0
	35	0;37, 30, 0, 0	0;37, 3, 31, 45 *
12	36	0;38, 34, 17, 9	0;38, 7, 3, 31**
	37	0;39, 38, 34, 17	0;39, 10, 35, 17**
	38	0;40, 42, 51, 26	0;40, 14, 7, 3
13	39	0;41, 47, 8, 34	0;41, 17, 38, 49
	40	0;42, 51, 25, 43	0;42, 21, 10, 35
	41	0;43, 55, 42, 51	0;43, 24, 42, 21
14	42	0;45, 0, 0, 0	0;44, 28, 14, 7
	43	0;46, 4, 17, 9	0;45, 31, 45, 52 *
	44	0;47, 8, 34, 17	0;46, 35, 17, 38
15	45	0;48, 12, 51, 26	0;47, 38, 49, 24
	46	0;49, 17, 8, 34	0;48, 42, 21, 10
	47	0;50, 21, 25, 43	0;49, 45, 52, 56*
16	48	0;51, 25, 42, 52	0;50, 49, 24, 42
	49	0;52, 30, 0, 0	0;51, 52, 56, 28
	50	0;53, 34, 17, 9	0;52, 56, 28, 14
17	51	0;54, 38, 34, 17	0;54, 0, 0, 0***
	52	0;55, 42, 51, 26	0;55, 3, 31, 45
	53	0;56, 47, 8, 34	0;56, 7, 3, 31*
18	54	0;57, 51, 25, 43	0;57, 10, 35, 17
	55	0;58, 55, 42, 52	0;58, 14, 7, 3
	56	1; 0, 0, 0, 0	0;59, 17, 38, 49
19	57	1; 1, 4, 17, 9	1; 0, 21, 10, 35
	58	1; 2, 8, 34, 17	1; 1, 24, 42, 21
	59	1; 3, 12, 51, 26	1; 2, 28, 14, 7
20	60	1; 4, 17, 8, 35	1; 3, 31, 45, 53

al-Farghānī, the entries having a single asterisk behind them have been truncated in the text rather than rounded. The two entries followed by two asterisks each have their final digits in the text one unit low, 6 and 34 respectively. Both are easily explainable as copyist's errors. The entry with three asterisks is 0;53,59,59 in the text.

The passage closes with an expression of Bīrūnī's desire to settle the ambiguity remaining in the meridian length determination by a survey himself. He notes an abortive attempt, presumably made during his short stay (about 1000) at the Gurgān court of Qābūs (Section 65). Dahistān (215:2, read Dihistān, the land of villages – dih is Persian for village) was the region just north of Gurgān and the Atrak River. It was the border district adjoining the flat deserts to the north (Le Str., p. 379).

(For general descriptions of medieval Islamic measurements of a degree along the meridian, see Scritti, pp. 408–457, and Comm. Vol., pp. 1–52).

70. Length of a Degree along an Oblique Great Circle (218:1 – 18)

This topic is indeed discussed in the third chapter (of Book 1) of the Geogr., but in a general way. The chapter is translated into German and its implications discussed in Mžik (pp. 19–20, 79–84) with the aid of the same figure which Bīrūnī introduces presently in the Taḥdīd (e.g. Figure 57). The English of 218:5 is necessarily somewhat ambiguous because the Arabic term samt does not have the identical connotation of English "azimuth". Abū Rayḥān does not mean that in proceeding, say, from E (on Figure C52) to H the surveyor marches on a path of constant azimuth. A trajectory of fixed azimuth on the sphere is, in general, a rhumb line and not a great circle. What is clearly intended here is the great circle arc EH.

Assuming as given $\varphi_H =$ YH, $\varphi_E =$ DE, $\Delta\Lambda$ = DY, and $\overline{\text{az}}.$ = AEL, the next passage (218:7–18) proceeds to show how the arc EH can be calculated. (In fact, this set of assumptions is redundant).

Precisely as was done in 207:10 we have

(218:11) $$\text{HK} = \text{arc Sin}\left(\frac{\text{Cos } \varphi_H \cdot \text{ Sin } \Delta\Lambda}{\text{R}}\right).$$

Then, by the Rule of Four in triangles HKE and ALE,

(218:12) $$\frac{\text{Sin HK}}{\text{Sin HE}} = \frac{\text{Sin (AL} = \overline{\text{az}}.)}{(\text{Sin LE}) = \text{R}}$$

So

$$\text{HE} = \text{arc Sin } (\text{R} \cdot \text{ Sin HK / Cos az.})$$

gives the arc HE in degrees. The same arc length having been determined in terrestrial units by the survey, the division of the second by the first gives the length of a degree on the earth's surface.

71. Radius of the Earth by Observation from a Mountain (218:19 – 221:2)

This passage presents an alternative method for determining the size of the earth. Essentially the same discussion appears in one of Bīrūnī's books on the astrolabe (Boilot RG 4; see also Wiedemann, and Scritti, p. 302).

In Figure C53, an adaptation of the cognate figure in the text, m and r are the height of the mountain and the radius of the earth respectively. From triangle ETK

$$\cos h = r / (r + m),$$

whence

(1) $$r = m \cos h / \text{vers } h = m \text{ Cos } h / \text{Vers } h.$$

where h is the angle of depression of the horizon.

Bīrūnī proceeds first to obtain two expressions which are equivalent to (1) above. From the similar triangles EZM and EKT

(219:11) $$\frac{\text{EZ } (= \text{R})}{\text{ZM } (= \text{Cos } h)} = \frac{\text{EK}}{\text{KT}}$$

and

(219:12) $$\frac{\text{EZ } (= \text{R})}{(\text{EZ} - \text{ZM}) = \text{Vers } h} = \frac{\text{EK}}{(\text{EK} - \text{KT}) = \text{EL}}$$

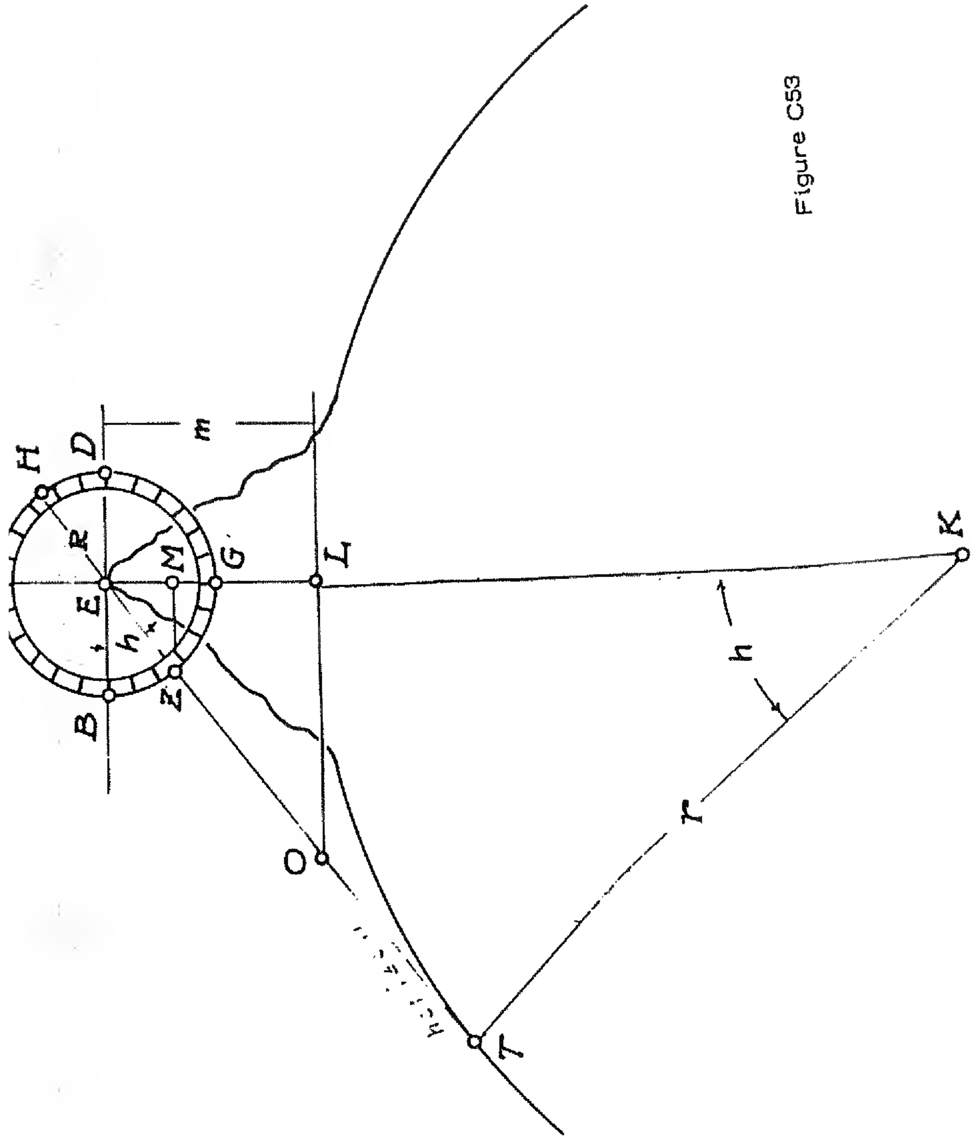


Figure C53

Solving for EK,

$$EK = m\left(\frac{R}{\mathrm{Vers}_R\ h}\right) = r + m .$$

Therefore

$$(2) \qquad r = m\left(\frac{R}{\mathrm{Vers}_R\ h}\right) - m .$$

For the second expression, he put

$$(219{:}17) \qquad \frac{EL}{LO} = \frac{\mathrm{Sin}\,(EOL = h)}{\mathrm{Sin}\,(OEL = \bar{h})}$$

and, solving for LO = OT,

$$LO = EL\,\frac{\mathrm{Cos}\ h}{\mathrm{Sin}\ h} = m\,\frac{\mathrm{Cos}\ h}{\mathrm{Sin}\ h} .$$

Bīrūnī says EO "can be found". It is, e.g., $m/\sin\ h = mR/\mathrm{Sin}\ h$. So $ET = EO + OT = \dfrac{mR}{\mathrm{Sin}\ h} + \dfrac{m\ \mathrm{Cos}\ h}{\mathrm{Sin}\ h}$. Line 219:19 should read $ET/KT = \mathrm{Sin}\ h/\mathrm{Cos}\ h$, whence

$$(3) \qquad r = KT = ET\left(\frac{\mathrm{Cos}\ h}{\mathrm{Sin}\ h}\right) = \frac{m}{\mathrm{Sin}\ h}\,(\mathrm{Cos}\ h + R)\,\frac{\mathrm{Cos}\ h}{\mathrm{Sin}\ h}$$

Figure 54, used to describe the work of Sanad (220:1 – 221:2), is confusing, partly because some of the letters in the text have been garbled, and partly because the circle containing B and Z is tacitly regarded as the celestial sphere, its radius so large that any point in the vicinity of K may be regarded as the center of the universe.

From the similarity of triangles ETK and BMH (not BME as in the translation, 220:11)

$$(220{:}12) \qquad MB / MH = EK / KT .$$

Hence, taking MB as R, the parameter of the trigonometric functions,

$$(220{:}13) \qquad \frac{MB\ (=R)}{(MB - MH) = \mathrm{Vers}\ h} = \frac{EK\ (= m + r)}{(EK - KT) = EL = m}$$

So

$$m + r = mR / \text{Vers } h, \quad \text{and}$$

$$r = \frac{mR}{\text{Vers } h} - m,$$

which is Bīrūnī's first expression above.

Concerning the Caliph Ma'mūn and his astronomer Sanad, see Section 20.

72. Finding the Height of a Mountain (221:3 – 222:9)

Our Figure C55 has proportions which are somewhat more realistic than the cognate drawing in the text. It is necessary to measure TA and GH (drawn heavily in the figure) in terms of the sides of the square board ABDG. Then from the similar triangles TAD and EDG,

$$(222{:}3) \qquad GE = AD \cdot DG / AT,$$

and from the similarity of triangles EZG and DHG,

$$m = EZ = GE \cdot GH / DG.$$

The weakness of the technique stems from the fact that the observer is attempting to measure, in effect, the small angle ADT, which is the parallax of the mountain peak E as viewed from the corners G and D of the square board. Bīrūnī does not give his findings for AT and GH, nor his computations. His result, shown to the equivalent of four sexagesimal digits in the next passage (at 223:2), cannot in fact be precise to more than two of these digits.

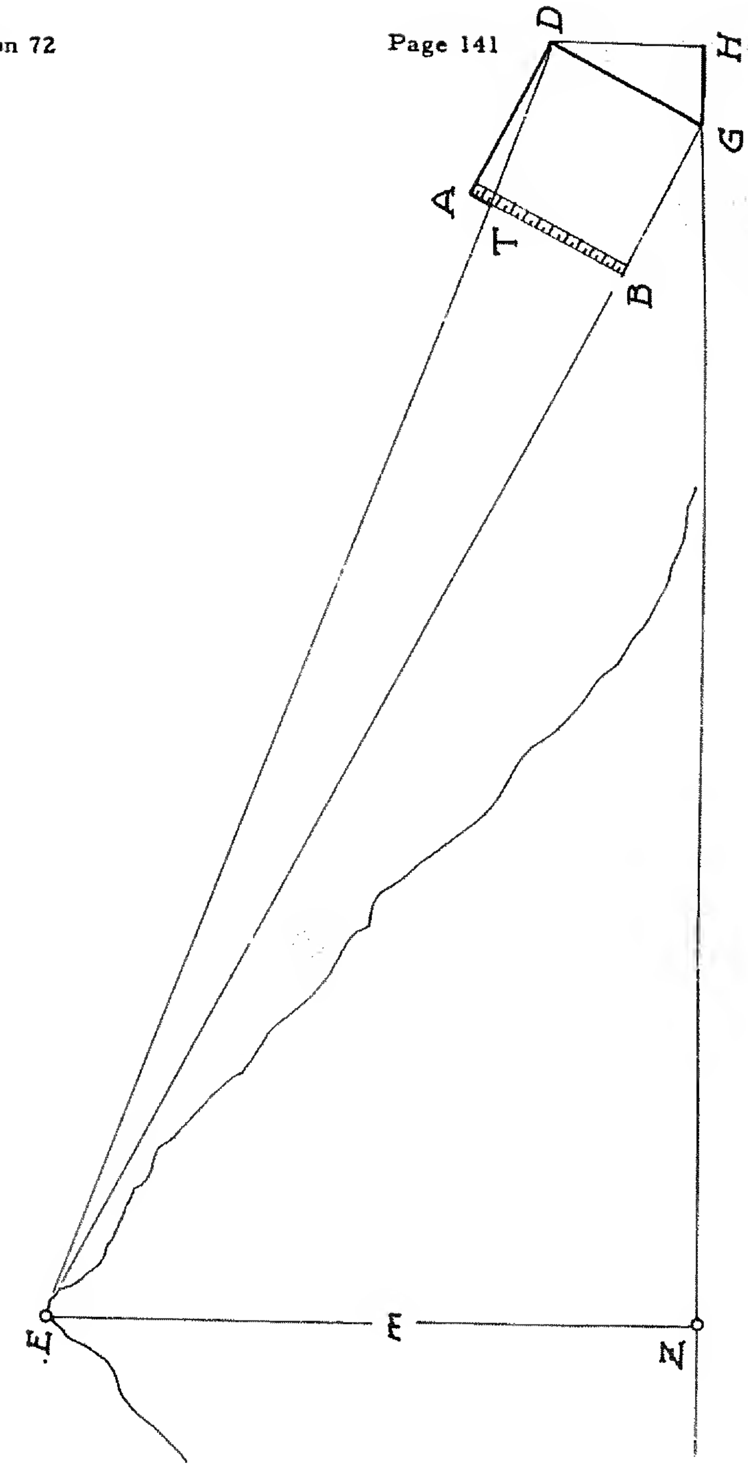


Figure C55

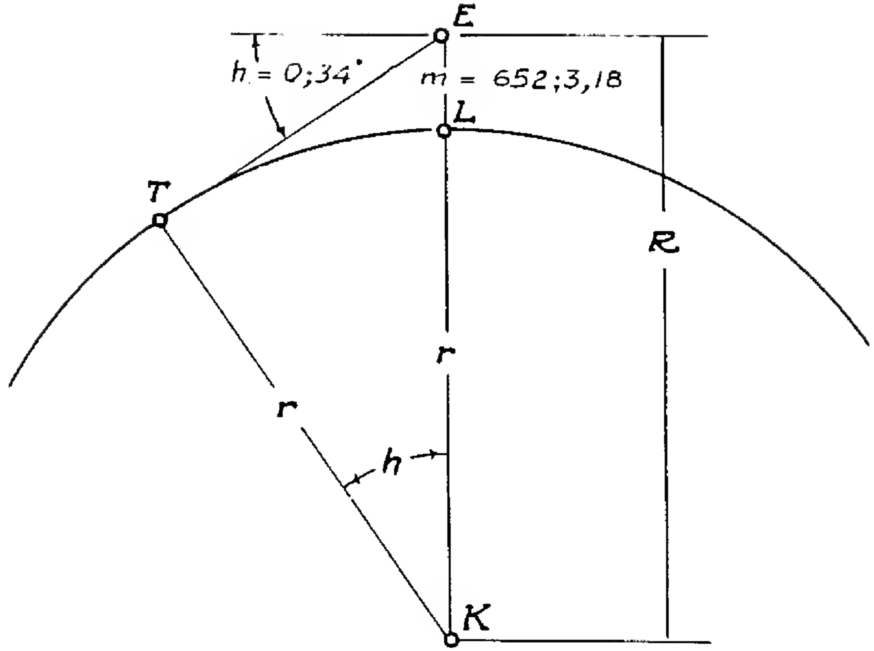


Figure C56

73. Bīrūnī's Observation at Nandana; Announcement of the Final Objective (222:10 – 226:9)

In this section Abū Rayḥan applies the techniques he has just explained to make an actual estimate of the size of the earth. The observations were made while he was living, probably in detention, at Nandana Fort during the year 1018 or thereabouts. The site is in northern Pakistan, and the installation there commanded the route by which Alexander the Great, Maḥmūd of Ghazna, and other conquerors penetrated the Indus valley (see Stein).

The computation itself is an application of our expression (1) in Section 71, never explicitly derived by Abū Rayḥān. Two numbers are needed, the height of the mountain, $m = 652;3,18$ (see Figure C56) and $h = 0;34^{\circ}$. The latter, reported to one sexagesimal digit, is at the limit of precision to be hoped for in the case of an angular distance measured with the instruments then available. Nevertheless, all available digits are carried along in m and in the succeeding computations.

It is illuminating to note that in the Canon (pp. 530–1) the same determination is repeated with the same h, but with the last sexagesimal digit (the 18) dropped from m. Successive results of the computations are shown in the three columns below, the last column being carried to a precision sufficient to guarantee two significant sexagesimal places in the result. The five percent error in the answer in the Taḥdīd is caused essentially by the use of trigonometric functions to three sexagesimal digits. The tables in the Canon are to four digits, but the value used for Cos h is badly off in the fourth digit, so that the result there is little better than the Taḥdīd's.

We note that in converting from an earth-radius in cubits to the equivalent miles along a degree of the meridian, Bīrūnī takes π as 22/7 and a mile of 4000 cubits.

	Taḥdīd	Canon	Accurate
Cos h	= 59;59,49	59;59,49,2	59;59,49,26,9
R – Cos h = Vers h	= 0;0,11	0;0,10,58	0;0,10,33,51
r in cubits = $\frac{m \cdot \cos h}{\text{Vers } h}$	= 12,803,337;2,9	12,851,369;50,42	13,331,000
miles/degree = $\frac{2\pi r}{360 \cdot 4000}$	= 55;53,15	56;5,50	58;33

(Cf. Scritti, pp. 302–305).

The chapter concludes with the statement that the object of the study is now the determination of the longitude of Ghazna. For this, terrestrial distances between many other localities will be employed, as well as the terrestrial latitudes previously discussed.

CHAPTER VI. RELATIONS BETWEEN DISTANCES, LONGITUDES, AND LATITUDES

74. Great Circle Distances and Geographical Coordinates (227:3 – 228:9)

Here three relations are derived – two special cases followed by the general one.

If two localities, A and B, have the same longitude and differing latitudes, then the great circle distance between them in miles is

(227:5) $$AB = k \cdot \Delta\varphi$$

if k is the number of miles per degree along a meridian.

On the other hand, if the latitudes are the same but the longitudes differ, the situation is as displayed in Figure C56.1. As Bīrūnī remarks in 227:10 the chord of the parallel of latitude joining A to B is identical with the chord of the great circle between these points. And by the similarity of the two isosceles triangles in the equatorial plane,

(227:12) $$\text{Crd}\ \widehat{AB} = \frac{\text{Crd}\ \Delta\Lambda \cdot \text{Cos}\ \varphi}{R},$$

and in miles

$$AB = k \cdot \text{arc Crd}\ \widehat{AB}.$$

For the general case, where both latitudes and longitudes differ, from the three similar isosceles triangles in Figure C57 the text obtains

(228:1) $$\text{Cos}\ \varphi_1 / \overline{\Delta\Lambda}_1 = R / \text{Crd}\ \Delta\Lambda = \text{Cos}\ \varphi_2 / \overline{\Delta\Lambda}_2$$

(where here the bar indicates the chord of a small-circle arc, not a complementary arc),

or

(228:3) $$\overline{\Delta\Lambda}_i = \text{Cos}\ \varphi_i\ \text{Crd}\ \Delta\Lambda / R, \quad i = 1,2.$$

The plane quadrilateral AZBD is an isosceles trapezoid, hence cyclic, and the theorem of Ptolemy applied in Section 47 above yields

(228:6) $$\overline{AB}^2 = (\text{Cos}\ \varphi_1 \cdot \text{Crd}\ \Delta\Lambda / R)\ (\text{Cos}\ \varphi_2 \cdot \text{Crd}\ \Delta\Lambda / R) + \text{Crd}^2\ \Delta\varphi,$$

or

$$\widehat{AB} = \text{arc Crd}\left[\left(\frac{\text{Cos}\ \varphi_1 . \text{Cos}\varphi_2 . \text{Crd}^2\Delta\Lambda}{R^2} + \text{Crd}^2\Delta\varphi\right)^{\frac{1}{2}}\right]$$

Multiplication of the above result, by k converts it into an expression for finding the great circle distance in miles between two localities in terms of their geographical coordinates, Bīrūnī does not claim to have discovered the algorism, but we have not found it in any other work.

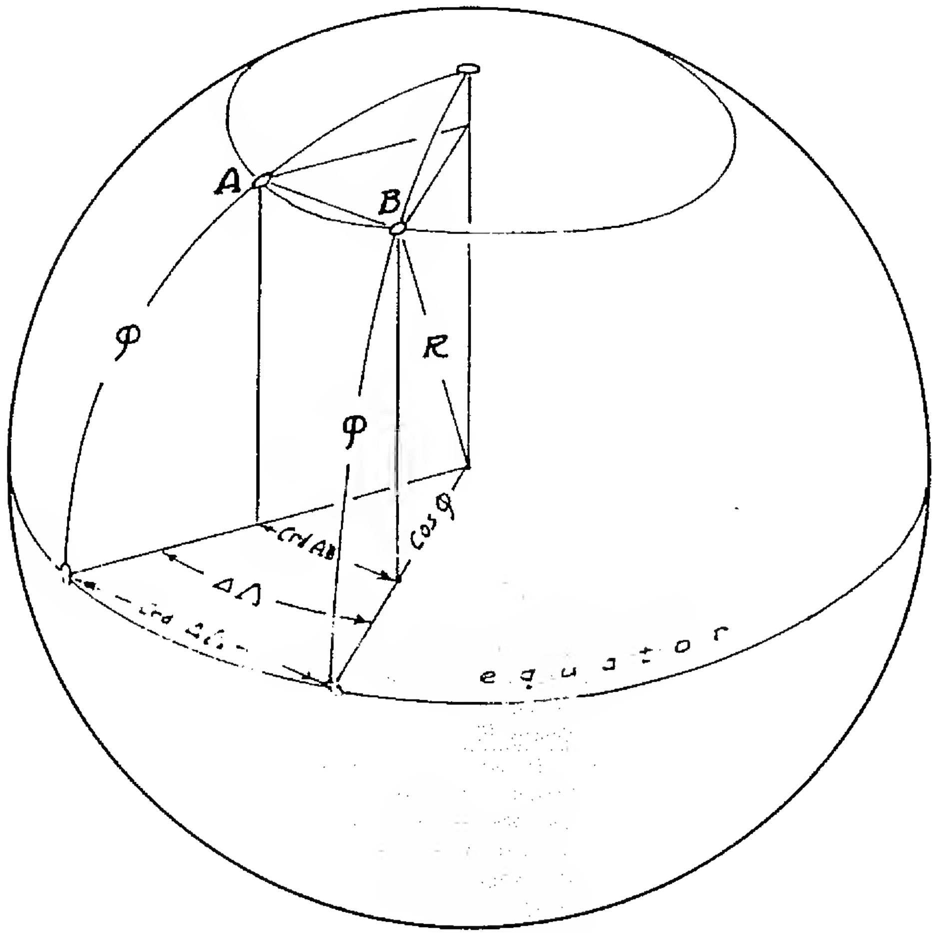


Figure C56.1

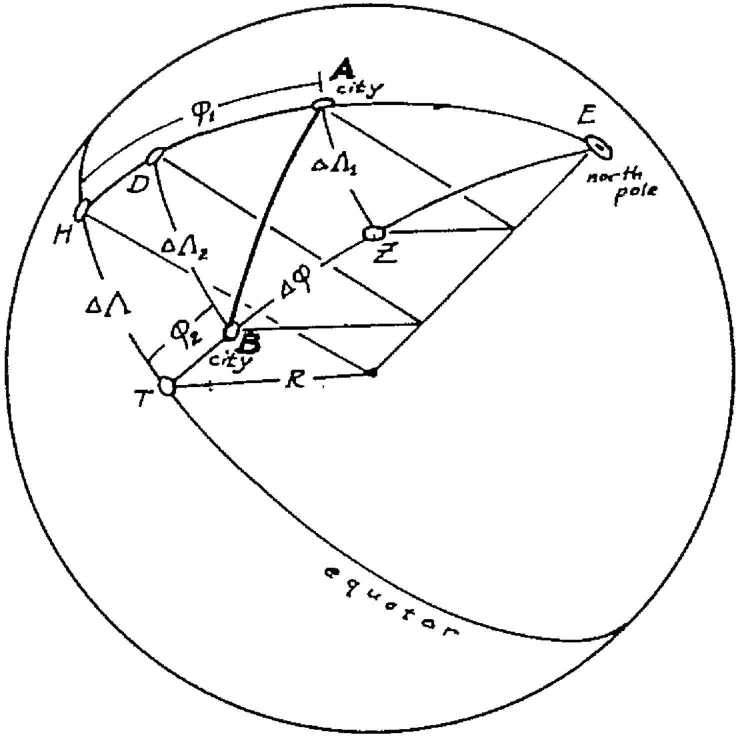


Figure C57

75. Indian Rules and Bīrūnī's Critique (228:10 – 234:11)

This passage is of interest because it contains odds and ends of information about Sanskrit astronomical works which had been put into Arabic. Most of the material is from a book conjectured to have been called the Bhūgolādhyāya (see Fazārī, p. 119), author and translator unknown. From it three rules are given for calculating terrestrial distance in terms of coordinates, rules cognate to the three discussed just above, and in the same order. Abū Rayḥān then explains and criticizes the rules.

A concept used in all three is the ṭawq al-madār (lit. the arch of the parallel circle), half the circumference of the latitude circle passing through a given locality. If c is the circumference of the earth, then the ṭawq is (see Figure C56.1)

(1) $$\tau = (c / 2) \cos \varphi .$$

The text gives a rule for evaluating the ṭawq. It is, in symbolic form,

(228:11) $$\tau = \frac{c}{2} - \frac{\text{Vers } \varphi}{2} \cdot \frac{c}{2} ,$$

provided that we read 228:14 as ". . . he subtracts the quotient from half a rotation (in farsakhs), which is 180 (in degrees)". The R is 3438' = 57;18, as Bīrūnī states later, in 230:9. This parameter is found in the Āryabhaṭīya (p. 19) and the Paitāmahasiddhānta.

The definition reduces to

$$\tau = \frac{c}{2} (1 - \text{vers } \varphi) = \frac{c}{2} \cdot \cos \varphi ,$$

which is our expression (1) above. The notion of the ṭawq occurs elsewhere in Bīrūnī's work, in the India (transl., vol. 1, p. 312), and in the Shadows (128:2–6, 221:1–18). The second passage in the latter has the same rule as that of the Taḥdīd. In the Shadows it is attributed to an anonymous zīj.

The value of $\frac{c}{2}$ of $3298 \frac{17}{25}$ ($= \frac{1}{2} \times 6597 \frac{9}{25}$) farsakhs comes ultimately from the Āryabhaṭīya (Fazārī, p. 121), as do the related 2100 farsakhs for the diameter of the earth and the approximation to π of

(229:15) $$3927 / 1250 = 3.1416 \approx 180 / 57;18$$

for

$$2100 \times 3.1416 = 6597 \frac{9}{25} .$$

This approximation indeed appears in al-Khwārizmī's Algebra (ed. p. 51, transl. pp. 71–72), but not explicitly in the extant version of his Zīj.

The same number appears as the ratios

$$\frac{125,664,000}{40,000,000} = \frac{125664}{40000} = 3.1416$$

The Great Sindhind, mentioned frequently in the literature, seems to have been compiled by Muḥammad b. Ibrāhīm al-Fazārī. But this passage is the only place where a Little Sindhind is named. We have no conjecture concerning its author. (Cf. Fazārī and Ya'qūb.)

As for the problem itself, the first special case is when $\varphi_1 = \varphi_2$ and $\Lambda_1 \neq \Lambda_2$. The rule claims that the distance

between the towns is

(228:16) $\widehat{AB} = \Delta\Lambda \cdot \tau / 180$

in farsakhs. Reference to expression (1) above and to Figure C56.1 is sufficient to demonstrate that what the Indian rule gives is the distance between the two localities along their common parallel of latitude, not the great circle distance. This is gone into in detail by Bīrūnī (230:17 – 231:14), who uses Figure 58 (see our C58) which shows both the great and small circle arcs, ABS and AOS respectively. By the Rule of Four applied to triangles KAH and KSM,

(231:8) $\text{Sin } KA / \text{Sin } KS = \text{Sin } AH / \text{Sin } SM.$

Now $KA < KS, \quad \therefore \quad AH < SM,$

and since $HA = MO, \quad \therefore \quad MO < SM.$

Hence the two arcs ASB and AOB differ, and since the former is a great circle it is shorter than the other.

After calculating the distance it is increased by a sixth of itself to allow for windings in the road.

The second case is for $\Lambda_1 = \Lambda_2$ and $\varphi_1 \neq \varphi_2$. The rule has

(229:4) $$\widehat{AB} = \frac{c}{4}\left(\frac{\Delta\Lambda}{90}\right) = 1649\frac{17}{50} \cdot \left(\frac{\Delta\Lambda}{90}\right).$$

This is correct, and Bīrūnī establishes its validity in 231:1–7. But now the coefficient for turns in the road is 5/4 instead of 7/6 as above.

For the general case, latitudes and longitudes both differing, the rule is

(229:6) $$\widehat{AB} = \left[\Delta^2\varphi + \left(\frac{\Lambda_1 \tau_1}{180} - \frac{\Lambda_2 \tau_2}{180}\right)^2\right]^{\frac{1}{2}},$$

where the subscript 1 is associated with locality A, and 2 with B. As Bīrūnī implies in 232:7 – 233:18, the algorism is an attempt to apply the Pythagorean proposition to the curvilinear triangle ABZ on Figure C60. But, even for small $\Delta\varphi$ and $\Delta\Lambda$, the approach is faulty. The arc AH (in farsakhs) is $\Lambda_1 \cdot \tau_1 / 180$, and BG is $\Lambda_2 \cdot \tau_2 / 180$, but their difference is not AZ. The error in assuming that it is will be of the same order of magnitude as AB itself unless G and H are close to the equator. Furthermore, the units of $\Delta\varphi$ are presumably degrees, not farsakhs. Perhaps some of the anomalies in the rule are from garbles introduced during the transmission from the Sanskrit source. But in its present form all Abū Rayḥān's criticisms are justified. Here the coefficient for windings is 4/3, for no apparent reason.

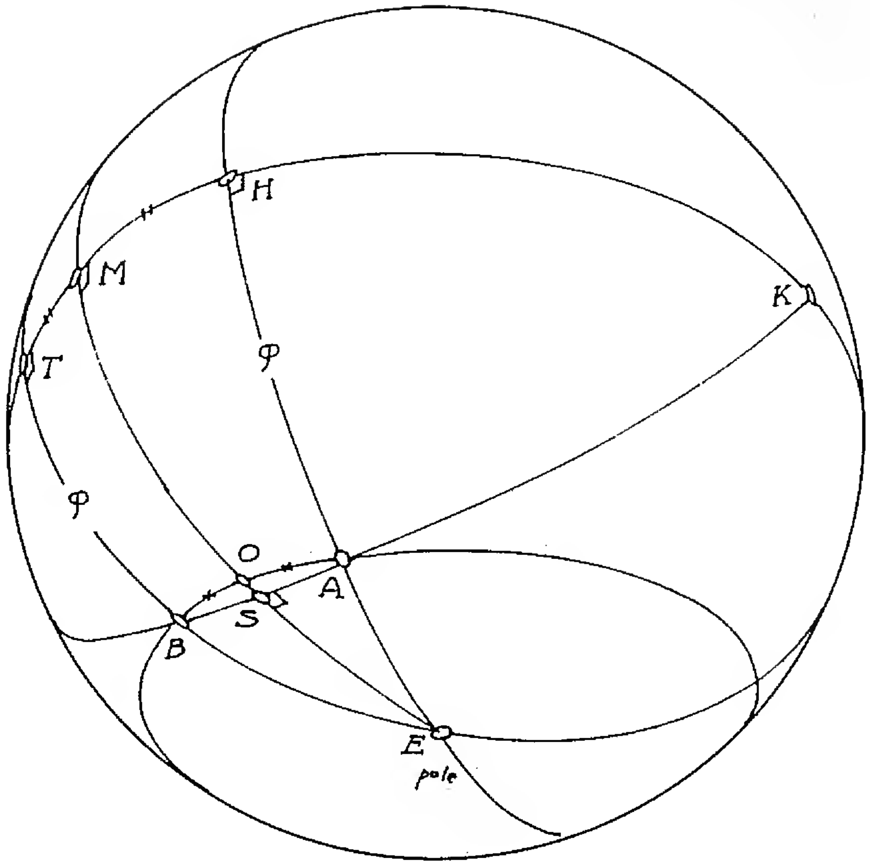


Figure C58

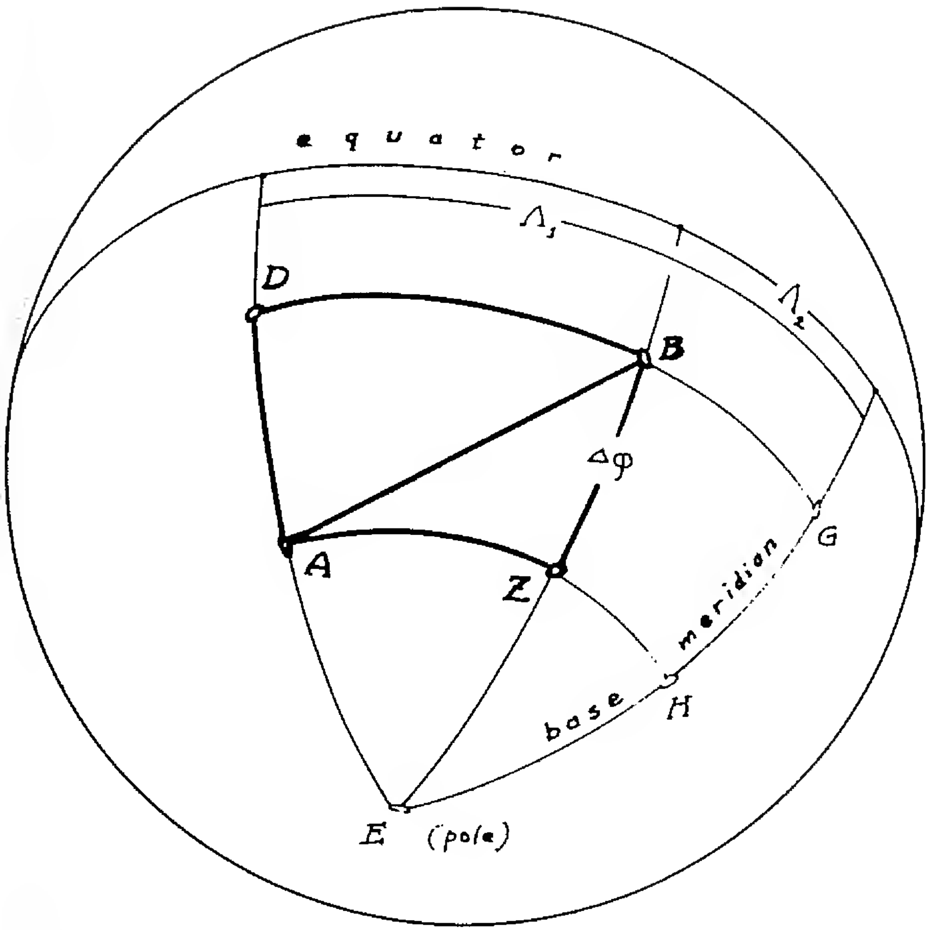


Figure C60

Pingree (in Fazārī, p. 117) has pointed out other geodetic rules of Indian origin which use the Pythagorean relation, but they are different from this one.

Marinus of Tyre (fl. 110 A.D.) worked out a system of projection in which parallels of latitude and meridians of longitude map as families of orthogonal straight lines. Their scales are so chosen that distances along all meridians are preserved, but only those along the latitude of Rhodes. Hence on Marinus' map latitudinal distances north of Rhodes are depicted as greater than they actually are, and those to the south as less. (See Exact Sc., p. 220).

The zīj of Battānī (vol. I, pp. 136, 318; III, p. 206, see also Section 22) contains a rule for calculating the azimuth of the qibla in which, as Bīrūnī says, a parallel of latitude has been treated as though it were a great circle.

76. Great Circle vs. Road Distance, Mecca to Baghdad (234:12-235:18

Presumably in order to examine the difference between actual distances by road and those calculated from known coordinates, Bīrūnī compares these two distances in the case of Mecca and Baghdad. The latitudes he takes as 21;40° and 33;25° respectively, as previously (in 210:1 and 100:9, etc.), and $\Delta\Lambda$ as 3°. Presumably by making use of the relation of 228:6 (see Section 74 above) he reports a great circle distance of

(234:13) $$\widehat{AB} = 12;1,51^\circ .$$

He violates his normal practise of presenting the entire computation, and states only the result. We obtain

$$\widehat{AB} = 12;2,44^\circ .$$

He multiplies the arc by 56;40 (see, e.g. 212:15) getting

$$AB = 681;44,50^\circ .$$

The Caliph al-Ma'mūn (Section 20) had the distance measured along the ground and was told that it is 712 miles. The difference, as Bīrūnī remarks (234:18), is of the order of a twenty-fourth of the whole.

CHAPTER VII. THE MAIN LONGITUDE COMPUTATION - THE NORTHERN TRAVERSE

77. The Longitude Difference Between Baghdad and Rayy (236:1 - 239:11)

All preliminaries being disposed of, the author proceeds to the determination of the longitude of Ghazna by successive calculations of the longitude differences between Baghdad, Rayy, Jurjāniya, and Balkh (235:10), in terms of their latitudes and the estimated great circle distances between them. To do so he again derives (237:9-15) by use of Figure 61 (or C57) the expression of 228:6 displayed in Section 74, this time solving for $\Delta\Lambda$ to obtain

$$(238{:}12)\quad \Delta\Lambda = \text{arc Crd}\sqrt{(\text{Crd}^2 AB - \text{Crd}^2 \Delta\varphi)\frac{\text{Cos }\varphi_2}{\text{Cos }\varphi_1}}\cdot\frac{R}{\text{Cos }\varphi_2}.$$

Note that Abū Rayḥān could have saved himself (and us) a long division operation each time the algorism is applied by writing the equivalent

$$\Delta\Lambda = \text{arc Crd}\sqrt{\frac{\text{Crd}^2 AB - \text{Crd}^2\Delta\varphi}{\text{Cos }\varphi_1\cdot\text{Cos }\varphi_2}}\cdot R \quad .$$

Evidently relations which are obvious when written in modern symbols are not apparent when expressed as verbal rules.

In particular, for Baghdad at A and Rayy at B, Bīrūnī takes

$$(237{:}4)\qquad AB = (\text{road distance})\frac{5}{6} = (158 \text{ farsakhs})\frac{5}{6}$$
$$= 158\times 3\times\frac{5}{6}\text{ miles} \quad 397 \text{ miles}$$
$$= 397\,/\,(56\tfrac{2}{3}\text{ miles / degree}) = 7;0,21^\circ$$

$$(237{:}16)\qquad \varphi_1 = 33;25^\circ \text{ (Baghdad)}$$

$$(238{:}2)\qquad \varphi_2 = 35;34,39^\circ$$ (Rayy; for al-Khujandī and al-Hirawī

see Sections 26 and 23 respectively). Hence

$$(238{:}5)\quad \Delta\Lambda = \text{arc Crd}\left(\sqrt{(7;13,54^2 - 2;15,45^2)48;47,59/50;4,52}\times 60/48;47,52\right)$$
$$= \text{arc Crd } 8;27,50$$
$$= 8;5,20^\circ .$$

This computation is precise to three sexagesimal places, except that in the following four instances

$$\text{Crd AB} = 7;19,54,56,$$
$$\text{Cos }\varphi_1 = 5;4,52,32,$$
$$\sqrt{\ } = 6;53,2,33,$$

and

$$\text{arc Crd }(-) = 8;5,20,50^\circ,$$

the accurate values shown above must be truncated, not rounded, to give the numbers in the text.

Accepting Bīrūnī's result and adding it to his Λ for Baghdad (239:6) we obtain for Rayy,

$$\Lambda = 70^\circ + \Delta = 78;5,20^\circ$$

In al-Khwārismī's geography this $\Delta\Lambda$ is indeed the five degrees which Bīrūnī attributes to "the zījes". However, in al-Battānī's zīj it is 6;0°, in Naṣīr al-Dīn al-Ṭūsī's it is 6;20°, and in the Shāmil Zīj it is 4;50° (<u>Geogr. Tables</u>). Since the actual difference is 7;1°, Bīrūnī's result is too large, but better than some of the other values.

The individual mentioned at 239:1 is the famous Muḥammad b. Zakarīya al-Rāzī (d. 932), known in the medieval West as Rhases. As a young man he left his native Rayy to study for a time in Baghdad. Bīrūnī compiled a bibliography of his works (<u>Suter</u>, p. 47; <u>Boilot</u>, p. 236).

78. Longitude Difference Between Rayy and Jurjaniya (240:1 - 14)

For this second application of expression 238:12 in Section 77 above,

$$\widehat{AB} = (185 \text{ farsakhs})\frac{5}{6} = (185\times 3 \text{ miles})\frac{5}{6}$$
$$\approx 463 \quad (56\tfrac{2}{3}\text{ miles / degree})$$

(240:8) $\quad = \; 8;10,14^{\circ}$, where again a sixth of the road distance is attributed to curves.

(238:4) $\quad \varphi_1 = 35;34,39^{\circ}$ (Rayy)

(240:3) $\quad \varphi_2 = 42;17^{\circ}$ (Jurjāniya)

So

$$\Delta\Lambda = \text{arc Crd}\left(\sqrt{(8;33,16^2 - 7;1,5^2)44;23,22/48;47,59} \times 60/44;23,22\right)$$

$$= \text{arc Crd } 6;18,20$$

(240:14) $\quad = 6;1,26^{\circ}$.

This computation has an error; to four significant digits

(240:8) $\quad \text{Crd } AB = 8;33,56,32$

and the text has the following values truncated rather than rounded:

(240:4) $\quad \text{Crd } \Delta\varphi = 7;1,5,58,$

$\quad \text{Crd } \varphi_2 = 44;23,22,36,$

and $\quad \text{arc Crd } (\text{---}) = 6;1,26,54^{\circ}$.

Adding Bīrūnī's result to the Λ obtained in the preceding section for Rayy, we have

$$\Lambda = 78;5,20^{\circ} + \Delta\Lambda = 78;5,20^{\circ} + 6;1,26^{\circ}$$

$$= 84;6,46^{\circ}$$

for Jurjāniya.

79. The Longitude of Jurjān from the Coordinates of Rayy and Jurjāniya (241:1 – 245:5)

Here Bīrūnī interrupts the main computation to investigate the location of Jurjān, which he (wrongly) assumes lies on the great circle between Rayy and Jurjāniya, the leg of the traverse just calculated. To do this he runs through a series of relations based on Figure C62, culminating in a value for $\Delta\Lambda_{BT}$, the longitude difference between Rayy and Jurjān.

Before this he discusses the great circle distance between the two places, naming routes through the following cities: Qūmis is the modern Dāmghān, about forty miles south of the southeast corner of the Caspian (LeStr., p. 364). Dunbāvand (now Damāvand) is a town at the foot of the mountain having the same name, about forty miles northeast of Rayy (Tehrān, LeStr., p. 371). Āmul and

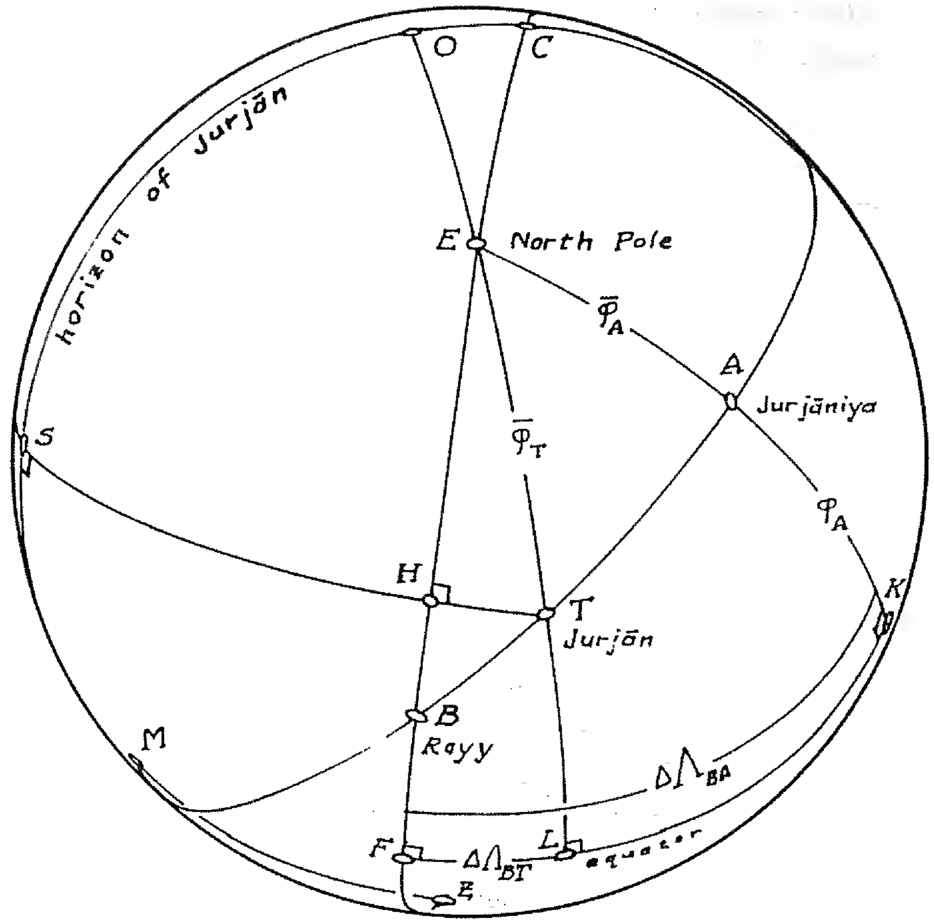


Figure C62

Sārya (modern Sārī), each at various times capital of the province of Ṭabaristān, are both on the Caspian littoral, Āmul slightly east of Rayy, Sārya still farther east (LeStr., p. 370).

(242:3) $\widehat{BT} = (70 \text{ farsakhs})\frac{5}{6} = (70 \times 3 \text{ miles})\frac{5}{6} = 175 \text{ miles}$

$= 175 / 56\frac{2}{3} = 3;5,18^{\circ}$.

By application of the sine law to the oblique triangle ABE,

(242:8) $\text{Sin } AB/\text{Sin } AE = \text{Sin} \angle BEA/\text{Sin} \angle ABE.$

Now

(240:7, 242:13) $\text{Sin } AB = \text{Sin } 8;10,14^{\circ} = 8;31,38,$

(240:10) $\text{Sin } AE = \text{Cos } \varphi_{A} = \text{Cos } 42;17^{\circ} = 44;23,22,$

(240:14) $\text{Sin } BEA = \text{Sin}\Delta\Lambda_{AB} = \text{Sin } 6;1,26^{\circ} = 6;17,48,$

so

(242:14) $\text{Sin} \angle ABE = 44;23,22 \times 6;17,48 / 8;31,38$

$= 32;46,41.$

Now, applying the sine law to the right triangle THB,

(242:14) $\text{Sin} \angle ABE/\text{Sin} (\angle THB=90^{\circ}) = \text{Sin } HT/\text{Sin}(BT = 3;5,18^{\circ}).$

So

(242:15) $TH = \text{arc Sin}(32;46,41 \times 3;13,57/R) = \text{arc Sin } 1;45,57$

$= 1;41,12^{\circ},$

and

(243:1) $HS = \overline{TH} = 88;18,48^{\circ}.$

By the Rule of Four applied to the triangles BZM and SZH,

(243:2) $\text{Sin } BZ/\text{Sin}(BM = \overline{\widehat{BT}}) = \text{Sin}(ZH = 90^{\circ})/\text{Sin } HS.$

So

$BZ = \text{arc Sin}(R \times \text{Cos } BT/\text{Sin } HS)$

$= \text{arc Sin}(R \times 59;54,46 / 59;58,26)$

$= \text{arc Sin } 59;56,20$

(243:7) $= 87;24,57^{\circ},$

and $BH = \overline{BZ} = 2;35,3^{\circ}.$

(243:8, 238:2) $HE = \overline{EB - BH} = \bar{\varphi}_{B} - BH$

$= 35;34,39^{\circ} - 2;35,3^{\circ}$

$= 51;50,18^{\circ}$

(243:9) $EC = \overline{HE} = 38;9,42^{\circ}.$

To four sexagesimal digits

$\text{Sin } EC = 37;4,22,37,$

which appears truncated in the text.

By applying the Rule of Four to the triangles ECO and HCS,

(243:10) $\text{Sin } EC/\text{Sin } EO = \text{Sin } (CH = 90^{\circ})/\text{Sin } HS.$

so $EO = \text{arc Sin } (\text{Sin } EC.\text{Sin } HS/R)$

$= \text{arc Sin}(37;4,22 \times 59;58,26/R)$

$= \text{arc Sin } 37;3,24$

$= 38;8,30,49^{\circ}$

to four sexagesimal places. The text has

(243:13) $EO = 38;8,33^{\circ},$

an error.

Now $OE = TL = \bar{\varphi}_{T}$, the latitude of Jurjān, and

$ET = \overline{OE} = \varphi_{T} = 51;51,27^{\circ}$, whence

(243:14) $\text{Sin } ET = 47;11,17.$

Applying the Rule of Four to triangles ETH and ELF,

(243:15) $\text{Sin}(ET = \bar{\varphi}_{T})/\text{Sin } HT = \text{Sin}(EL = 90^{\circ})/\text{Sin}(LF = \Delta\Lambda_{BT}),$

so

$\Delta\Lambda_{BT} = \text{arc Sin}(R \cdot \text{Sin } HT / \text{Cos } \varphi_{T})$

(243:1) $= \text{arc Sin}(R \times 1;45,57/47;11,17)$

$= \text{arc Sin } 2;14,43$

(243:17) $= 2;8,41^{\circ}.$

Concerning Avicenna and Zarrayn Kīs, see Section 65. The former's determination for the longitude of Jurjān is

(244:1) $79;20^{\circ}$

Bīrūnī's is the longitude of Rayy (239:8) plus the difference just calculated:

(243:18) $78;5,20^{\circ} + 2;8,41^{\circ} = 80;14,1^{\circ}$

The latitude values for Jurjān here reported are

Bīrūnī (from 243:14)		$38;8,33^\circ$
Avicenna (244:3)		37°
al-Hirawī (245:3),	March, 982,	$38;0^\circ$
	March, 983,	$37,40^\circ$

80. The Longitude Difference Between Būshkānz and Jurjāniya (246:1 – 15)

Chapter X of the translation presents a long and involved series of computations yielding eventually the difference in longitude between Jurjāniya and Kāth, the city of Khwārazm. The longitude of Kāth being independently known, the result gives a check on the longitude of Jurjāniya arrived at in Section 78. We break up the computation into the five parts listed below, and devote a section to the description of each part:

Part 1 calculates $\Delta\Lambda$ for Būskanz-Jurjāniya,
Part 2, beginning at 246:16, calculates CL in Figure C63.
Part 3, beginning at 248:1, calculates KC in the same figure.
Part 4, beginning at 248:12, obtains an independent value for the latitude of Kāth.
Part 5, from 249:12 on, completes the determination of $\Delta\Lambda$ for Jurjāniya-Kāth.

We proceed with Part 1, which is a straightforward application of the trapezoid algorism of Section 77, expression 238:12, and Figure C57. The distance Būshkānz-Jurjāniya is

(246:8) $\widehat{AB} = 17 \text{ farsakhs} = 3 \times 17 \text{ miles} = (51 / 56\tfrac{2}{3})$ degre
$= 0;54^\circ$,

where presumably the road is flat and straight, hence taken to be a great circle arc. The work at Būshkānz has already been discussed, in Section 29.

(246:5, 79:8) $\varphi_1 = 41;36^\circ$ (Būshkānz),

(240:10) $\varphi_2 = 42;17^\circ$ (Jurjāniya).

So

(246:9) $\Delta\Lambda = \text{arc Crd } (\sqrt{(0;56,33^2 - 0;42,56^2)44;23,22/44,52,4}$
$\times 60/44;23,22)$
$= \text{arc Crd } 0;49,28$
$= 0;47,14^\circ$

The computation as such is precise, but the value of $\text{Cos } \varphi_2 = 44;23,22$ from 240:10 has been truncated, as remarked in Section 78.

81. Azimuth Difference at Jurjāniya Between Kāth and Būshānz (246:16 – 247:19)

Part 2 also is an application of the trapezoid algorism of Section 77, but instead of being based on the equatorial system, the fundamental circle is the horizon of Jurjāniya and the curvilinear trapezoid is BOGY in Fig. C63. The result of the computation is not a $\Delta\Lambda$, but CL, the difference in the azimuths of Būshkānz and Kāth as reckoned from Jurjāniya.

In particular, corresponding to AB in expression 238:12 is

(247:1) $\widehat{BG} = 3 \text{ farsakhs} = 9 \text{ miles} = 0;9,32^\circ$.

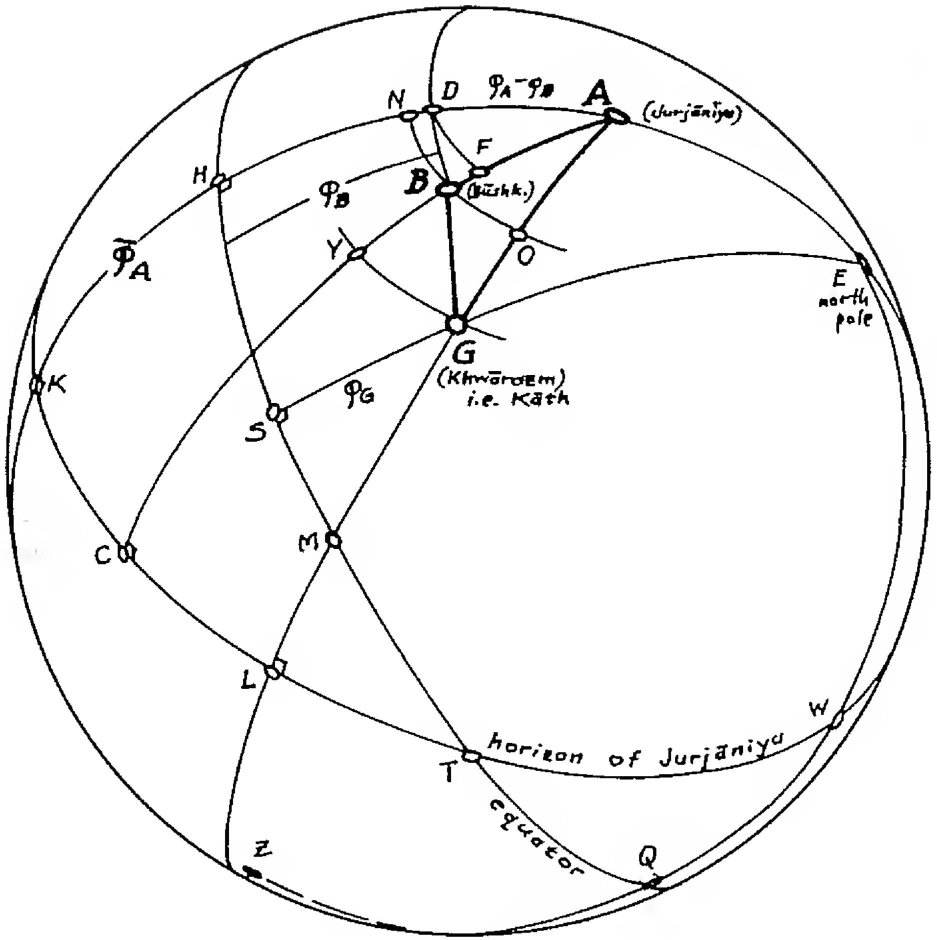


Figure C63

Corresponding to φ_1 is LG = $\overline{AG}$, and

(246:18) $\widehat{AG}$ = 19 farsakhs = 57 miles = $1;0,21^{\circ}$.

Corresponding to φ_2 is CB = $\overline{BA}$, and

(246:8) $\widehat{BA} = 0;54^{\circ}$.

Corresponding to $\Delta\varphi$ is CB − LG = OG = AG − AB = $1;0,21^{\circ} - 0;54,0^{\circ} = 0;6,20^{\circ}$.

(247:10) $CL = \text{arc Crd}\,(\sqrt{(\text{Crd}^2 BG - \text{Crd}^2 OG)\,\text{Sin}\,AB/\text{Sin}\,AG} \cdot R/\text{Sin}\,AB)$

$$= \text{arc Crd}\,(\sqrt{(\text{Crd}^2\,0;9,32 - \text{Crd}^2\,0;6,21^{\circ})\,\text{Sin}\,0;54^{\circ}/\text{Sin}\,1;0,21^{\circ}} \cdot R/\text{Sin}\,0;54^{\circ})$$

(248:12) $= \text{arc Crd}\,(\sqrt{0;9,59^2 - 0;6,39^2)0;56,33/1;3,12} \cdot R/ \cdot 0;56,33)$

$$= \text{arc Crd}\,6;40,36$$

$$= 6;22,45^{\circ}.$$

This result is in error. Disregarding the truncation of the root, which, to the three digits, is 0;7,2,36 instead of the 0;7,2 given in 247:16, the correct final quotient should have been

$$\text{Crd CL} = 7;35,10$$

instead of the 6;40,36 given at 247:19.

82. The Azimuth of Būshkānz from Jurjāniya (248:1−12)

Part 3 is a second application of the trapezoid algorism, based on the horizon of Jurjāniya. This time the trapezoid is NDFB on Figure C63, and the result corresponding to $\Delta\lambda$ is the arc KC on the horizon circle. Here the situation is complicated by the fact that the diagonal of the trapezoid, DB, is not a great circle arc, but a parallel of latitude. This does not vitiate the procedure, however, for expression 238:12 of Section 77 demands the chord of the diagonal, and the chord is the same, regardless of whether its arc is that of a great or a small circle.

In fact the rectilinear distance from B to D is

(248:2, 246:15) $BD = \text{Crd}\,\widehat{HS}\cos\varphi_B = \text{Crd HS Cos}\,\varphi_B/R$

$$= \text{Crd}\,0;47,14^{\circ}\ \text{Cos}\,41;36^{\circ}/R$$

(246:12) $= 0;49,28 \times 44;52,4/R$

$= 0;36,59.$

There is an error in the text; 248:3 gives this result as 0;36,51.

Corresponding to φ_1 in the algorism is BC $= 90^o - 0;54^o = 89;6^o$, the complement of AB, the "first distance" (246:18, 247:3).

Corresponding to φ_2 is KD $= \overline{DA} = 90^o - (\varphi_A - \varphi_B) = 0;41^o = 89:19^o$, from 246:6.

Hence, corresponding to $\Delta\varphi = \varphi_2 - \varphi_1$ is

(248:5) $89;19^o - 89;6^o = 0;13^o =$ BF.

So

(248:6) $\widehat{KC} = \text{arc Crd}\left(\sqrt{((BD)^2 - \text{Crd}^2 0;13) \text{Sin } 0;41^o/\text{Sin } 0;54^o \times R/\text{Sin } 0;42^o}\right),$

$= \text{arc Crd}\left(\sqrt{0;19,32,30,32 \times 0;42,56/0;56,33 \times R/0;42,56}\right)$

$= \text{arc Crd } 41;39,36$

$= 40;37,42^o.$

In this calculation the square root has been truncated; all other partial results are precise to three digits.

83. Calculation of the Latitude of Kāth (248:13 – 249:11)

Now, for Part 4,

KL $=$ KC $+$ CL $= 40;37,42^o + 6;22,45^o$

(248:12, 247:19) $= 47;0,27^o,$

and

(248:13) LT $= \overline{KL} = 42;59,33^o.$

The great circle arc EWQ has been drawn so that TW $=$ KL. Hence LT $+$ TW $=$ LT $+$ KL $= 90^o$. Apply the sine law to triangle TQW to obtain

(248:15) $\text{Sin}(TW = KL)/\text{Sin } WQ = \text{Sin}(\angle Q = 90^o)/\text{Sin}(\angle T = KH).$

Hence

$\text{Sin } WQ = \text{Sin } KL \cdot \text{Sin } (KH = \bar{\varphi}_A)/R$

(240:10) $= \text{Sin } 47;0,27^o \cdot \text{Cos } 42;17^o/R$

(248:13, 240:10) $= 43;54,12 \times 44;23,22/R$

$= 32;28,51$

So WQ $= 32;46,31^o$

and

(248:18) QZ $= \angle M = \overline{WQ} = 57;13,29^o$

There is an error in this computation, for Sin KL is in fact 43;53,11,41.

The above relation follows from the fact that W is the pole of ZLM, and M is the pole of ZQW.

Now apply the sine law to triangle LMT to obtain

(249:23) $\text{Sin } LM/\text{Sin } (\angle T = \bar{\varphi}_A) = \text{Sin } LT/\text{Sin } \angle M$

Hence

(248:20) LM $= \text{arc Sin}(\text{Sin } 42;59,33^o \cdot \text{Cos } 42;17^o/\text{Sin } 57;13,$

(248:14, 240:10) $= \text{arc Sin}(40;54,41 \times 44;23,22 / 50;26,53)$

$= \text{arc Sin } 35;59,53$

$= 36;51,3^o$

The final operation has an error; the result should have been $36;52,10^o$.

Now

(249:2) MG $=$ GL $-$ ML $= \overline{AG} -$ ML

$= 90^o - 1;0,21 - 36;51,3^o$

$= 52;8,36^o$

By a third application of the law of sines, this time to triangle MGS,

(249:3) $\text{Sin } MG/\text{Sin}(GS = \varphi_G) = \text{Sin}(\angle S = 90^o)/\text{Sin } \angle M.$

So

(249:5) $\varphi_G = \text{arc Sin}(\text{Sin } 52;8,36^o \cdot \text{Sin } 57;13,29^o/R)$

(248:19) $= \text{arc Sin }(47;22,22 \times 50;26,53 / R)$

$= \text{arc Sin } 39;49,52$

(249:6) $= 41;35,40,$

the latitude of Kāth, the city of Khwārazm.

Bīrūnī cites also a result obtained c.990 A.D., when he was seventeen years old, by direct observation with a ring:

(249:11) $\bar{\varphi}_G = 48;30^\circ$

whence $\varphi_G = 41;30^\circ$.

84. The Longitude Difference Between Kath and Jurjāniya (249:12 – 250:19)

Part 5 is a straightforward application of the trapezoid algorism of Section 77 to obtain the $\Delta\Lambda$ between Kāth, the city of Khwārizm, and Jurjāniya.

The great circle distance corresponding to AB in expression 238:12 is now, on Figure C83,

(246:18) $AG = 1;0,21^\circ$.

(249:6) $\varphi_1 = 41;35,40^\circ$ (Kāth)

(240:3) $\varphi_2 = 42;17^\circ$ (Jurjāniya)

(250:2) $\Delta\varphi = 0;41,20^\circ$.

So

$$\Delta_1\Lambda = \text{arc Crd}\left(\sqrt{(\text{Crd}^2 1;0,21^\circ - \text{Crd}^2 0;41,20^\circ)\text{Cos } 42;17^\circ/\text{Cos } 41;35,40^\circ \cdot R/\text{Cos } 42;17^\circ}\right)$$

(240:10) $$= \text{arc Crd}\left(\sqrt{(1;3,11^2 - 0;43,17^2)44;23,22/44;52,11 \times R/44;52,11}\right)$$

$$= \text{arc Crd } 1;1,53$$

$$= 0;59,6^\circ .$$

There are two errors in the computation. Correct values are

Crd 1;0,21 = 1;4,15, not 1;3,11,

and Cos 41;35,40 = 44;52,18, not 44;52,11.

To use this we need a value for the longitude of Kāth. Bīrūnī observed there the lunar eclipse of 24 May, 997 (250;11, <u>Oppolzer</u> No. 3403) while Abū al-Wafā (see Section 24) observed it at Baghdad. They concluded that there was an hour's difference in the local times of this event, corresponding to a $\Delta\Lambda$ of $(1/24)360^\circ = 15^\circ$. This should be added to the longitude of Baghdad, 70° (239:6), to give

(250:15) $\Lambda = 70^\circ + 15^\circ = 85^\circ$.

Jurjāniya being west of Kāth, the longitude of the former will be

$\Lambda = 85^\circ - \Delta\Lambda = 85^\circ - 0;59,6^\circ$

(250:16) $= 84;0,54^\circ$.

Of course, the precision implied by the three significant digits is illusory. But the result is close to the

(250:18) $\Lambda = 84;6,46^\circ$

obtained in Section 78.

85. The Longitude Difference Between Jurjāniya and Balkh (251:1 – 252:11)

The author now returns to the main traverse, interrupted at Section 80, and calculates $\Delta\Lambda$ from Balkh to Jurjāniya.

In expression 238:12 of Section 77 arc AB is now the great circle distance between the two cities. It is taken as

$$\widehat{AB} = (150 \text{ farsakhs})(1 - \tfrac{1}{3} \cdot \tfrac{1}{5}) = (140 \times 3) \text{ miles}$$

(251:12) $= 7;24,42^\circ$

The road length here estimated spans flat desert for most of its course, and lies pretty much along a great circle. The crossing of the Oxus at Kālif (<u>LeStr</u>., p. 442) was secured by fortifications on both banks. Bīrūnī thought this point marked a change of direction, hence his coefficient of 14/15. In any case, the result is quite good.

(251:4) $\varphi_1 = 36;41,36^\circ$ (Balkh. For Sulaimān and his observations there, see Section 23).

(240:3) $\varphi_2 = 42;17^\circ$ (Jurjāniya)

(251:5) $\Delta\varphi = 5;35,24^\circ$.

Hence

(251:14, 240:10) $$\Delta\Lambda = \text{arc Crd}\left(\sqrt{(7;45,22^2 - 5;51,5^2)44;23,22/48;6,38 \cdot R/44;23,22}\right)$$

$$= \text{arc Crd } 6;36,25$$

(252:4) $$= 6;18,54^\circ ,$$

and for Balkh

(250:16) $\Lambda = \Lambda_{\text{Jurjāniya}} + \Delta\Lambda = 84;0,54^\circ + 6;18,54^\circ$

(252:5) $= 90;19,48^\circ$.

There are no serious errors in the computation. The cosines of φ_1 and φ_2 have been truncated rather than rounded, otherwise the results are precise to three digits.

86. The Coordinates of Darghān from those of Jurjāniya and Balkh (253:1 – 255:11)

This is another digression from the main traverse, to fill in coordinates for an intermediate locality. The calculations with which the passage begins can be justified as follows.

The theorem of Ptolemy on cyclic quadrilaterals (Section 47) applied to the rectilinear trapezoid ADBZ of Figures C57 and 61 gives

$$AB^2 = AD^2 + AZ \cdot DB,$$

or

$$AZ \cdot DB = AB^2 - AD^2. \tag{1}$$

Moreover $\quad DB/AZ = \operatorname{Cos}\varphi_1/\operatorname{Cos}\varphi_2,$

provided that in Figure C57 φ_1 and φ_2 are switched, as are DB and AZ, in order to make the configuration the same as that of Figure 61. Substituting,

$$AZ\left(AZ\,\frac{\operatorname{Cos}\varphi_1}{\operatorname{Cos}\varphi_2}\right) = AB^2 - AD^2,$$

or

$$AZ = \sqrt{(AB^2 - AD^2)\operatorname{Cos}\varphi_2/\operatorname{Cos}\varphi_1}$$

$$= \sqrt{(\mathrm{Crd}^2 AB - \mathrm{Crd}^2\,\Delta\varphi)\operatorname{Cos}\varphi_2/\operatorname{Cos}\varphi_1}$$

This demonstrates the truth of Bīrūnī's remark (252:5), that the root in expression (238:12) is the chord AZ.

Now, solving expression (1) above for DB,

$$(253:6)\qquad DB = \frac{AB^2 - AD^2}{AZ} = \frac{\mathrm{Crd}^2 AB - \mathrm{Crd}^2\,\Delta\varphi}{\sqrt{\quad}}$$

$$(251:13, 252:1) = \frac{25;35,6,37,39}{4;53,24}$$

$$(253:6)\qquad = 5;13,1.$$

Bīrūnī has 5;18,1. Apparently he forgot to leave the tail off the jīm (=3), which is the convention in Arabic alphabetical sexagesimals, hence later read it as a ḥa' (=8).

Turning to Figure C64 we note that Bīrūnī makes the tacit assumption that Darghān, the coordinates of which are to be determined, lies on the great circle joining Jurjāniya to Balkh. Darghān, on the west bank of the Oxus, was the first great city on the highroad from Kāth to Balkh (LeStr., p. 451).

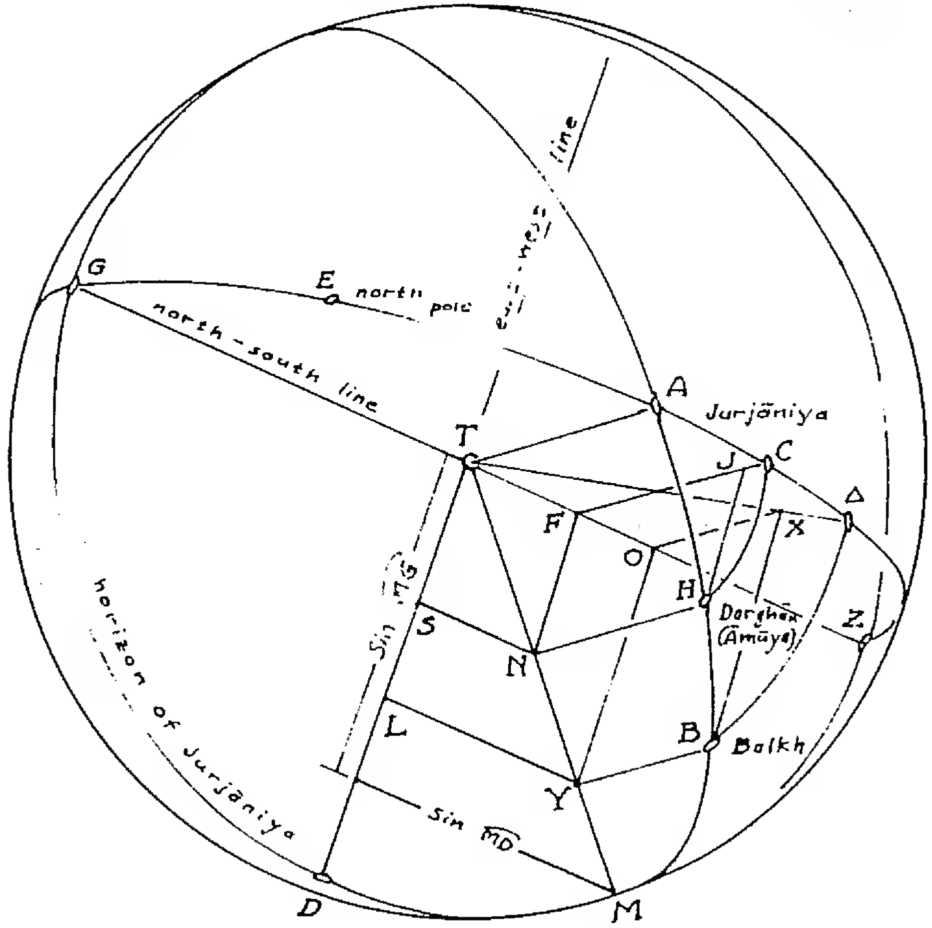


Figure C64

On Figure C64 B is the projection of Balkh on the celestial sphere, whence BM is its altitude with respect to the horizon of Jurjāniya. The complement of $\widehat{BM}$ is $\widehat{AB}$, which is $7;24,42^{\circ}$, from 251:12, and

(253:15) $\quad \text{Sin } AB = 7;44,23 = TY.$

Let $\widehat{B\Delta}$ (not shown on Figure 64) be the parallel of latitude through B. It will then be the same arc as DB in Figure 61, the chord of which was just calculated in 253:6. Accepting Bīrūnī's incorrect value, the corresponding great circle arc will be

(253:7) $\quad \text{arc Crd } 5;18,1 = 5;3,47^{\circ}$

Then BX (not shown on Figure 64) drawn perpendicular to $T\Delta$, will be

(253:9) $\quad BX = \text{Sin } 5;3,47^{\circ} = 5;17,42,26$

(253:15) $\quad = YO.$

(Bīrūnī wrongly writes the third digit as 43).

Now apply the Pythagorean theorem to triangle TOY to obtain

(254:1) $\quad YL = TO = \sqrt{TY^2 - YO^2} = \sqrt{7;44,23^2 - 5;17,43^2}$

$\quad = 5;38,24.$

By similar right triangles,

(254:2) $\quad TY/YL = (TM = R)/\text{Sin } MD,$

so $\quad MD = \text{arc Sin}(YL \cdot R/TY) = \text{arc Sin}(5;38,24 \times R/7;44,23)$

$\quad = \text{arc Sin } 43;43,21$

(254:5) $\quad = 46;46,42^{\circ}.$

So Sin GM = Sin $\overline{MD}$ = Sin $43;13,18^{\circ}$ = 41;5,22. (In Figure C64 GM is not the complement of MD, but 90° + MD, and the expression above holds).

Also, by similar right triangles,

(254:6) $\quad TY/YO = (TM=R)/\text{Sin } GM$

So

(253:15) $\quad \text{Sin } GM = YO \times R/TY = 5;17,43 \times R/7;44,23$

$\quad = 41;5,20.$

No motive is given for this double determination of the same quantity. The value shown restored in the translation is correct. The text has 41;5,22 as in 254:5.

The great circle distance from Jurjāniya to Darghān is taken as

(254:10) $\quad \widehat{AH} = 50 \text{ farsakhs} = 50 \times 3 \text{ miles}$

$\quad = (150 \;/\; 56\tfrac{2}{3})^{\circ}$

$\quad = 2;38,49^{\circ}$

Apparently the phrase translated as "to shorten the long unit" indicates a simple conversion with no coefficient to compensate for windings in the road.

(254:12) $\quad TN = \text{Sin } AH = 2;46,15.$

By similar right triangles

(254:13) $\quad TN/FN = (TM = R)/\text{Sin } MG.$

So

(254:5) $\quad NF = TN \cdot \text{Sin } MG/R = 2;46,15 \times 41;5,22/R$

(254:14) $\quad = 1;53,51.$

As the author remarks, HC being on the parallel of latitude through Darghān,

$$NF = \text{Sin}_{\rho}\, HC,$$

it being understood that ρ is the radius of that parallel. (This radius appears nowhere on Figures 64 or C64).

By the Pythagorean theorem

(254:17) $\quad \text{Sin } AC = TF = \sqrt{TN^2 - NF^2}$

$\quad = \sqrt{2;46,15^2 - 1;53,51^2}$

$\quad = 2;1,9.$

The text states that

$$\widehat{AC} = \text{arc Sin } 2;1,9 = 1;46,43^{\circ}.$$

This is in error; the correct value is $1;55,39^{\circ}$. Using the erron[e] result,

(254:18) $\quad \bar{\varphi}_H = \bar{\varphi}_C = \widehat{EC} = \widehat{AC} + \widehat{EA} = \widehat{EH}$

(240:3) $\quad = AC + \bar{\varphi}_A = 1;46,43^{\circ} + 47;43^{\circ}$

$\quad = 49;29,43^{\circ}.$

So the latitude of Darghān is

(255:2) $\varphi_C = 40;30,17^\circ = \varphi_H$.

Further, by similar triangles

(255:3) $\left[(\sin \widehat{EH} = \bar{\varphi}_H) = \rho\right] / \left[(\sin_\rho \widehat{HC}) = HJ = NF\right]$

$= R/\left[(\sin \Delta\Lambda_{HA}) = KP\right]$.

The validity of this may be inferred from Figure C64.1, where A as Jurjāniya and H as Darghān are disposed as in Figure C64, but where the equator, the parallels of latitude through A and H, and their respective planes are indicated. HJ and KP are perpendicular to CV and PT respectively.

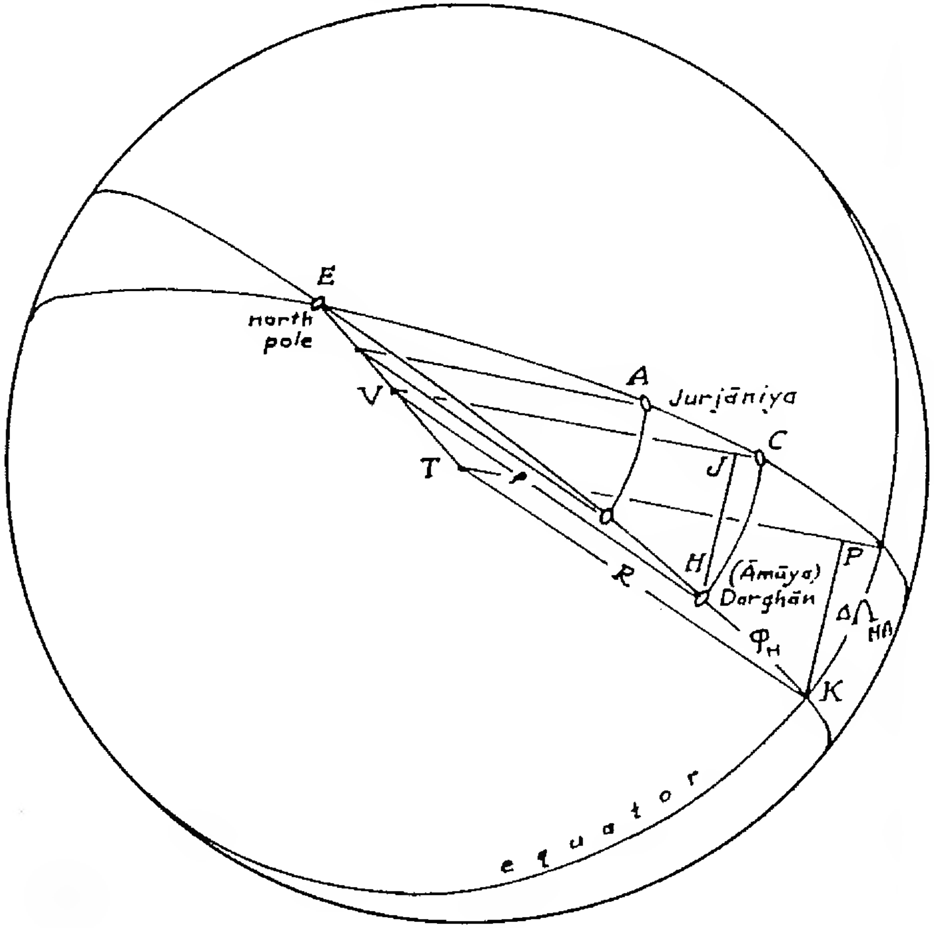


Figure C64.1

Hence

(255:6) $\sin \Delta\Lambda_{HA} = NF \cdot R/\sin \bar{\varphi}_H$

$= 1;53,51 \times R/\sin 49;29,43^\circ$

(255:3) $= 1;53,51 \times R/45;37,17.$

$= 2;29,44.$

In this calculation the last digit of $\sin \bar{\varphi}_H$ should be 16, not 17 as in the text. From computations later in the text, at 258:19, it is clear that the author intended a 16.

(255:7) $\Delta\Lambda_{HA} = \text{arc} \sin 2;29,44 = 2;23,2^\circ$,

and the longitude of Darghān is

$\Lambda = \Lambda_{\text{Jurjāniya}} + \Delta\Lambda_{HA}$

(250:16) $= 84;0,54^\circ + 2;23,2^\circ$

(255:7) $= 86;23,56^\circ$.

87. The Coordinates of Āmūya from those of Balkh and Jurjāniya (256:1-16).

In the same manner as he has just done with Darghān, Bīrūnī now assumes that Āmūya (see Section 8) lies on the great circle between Jurjāniya and Balkh, and he uses this and its distance from the former to calculate its latitude and longitude. Figures C64 and C64.1 again serve for the derivation where now point H represents Āmūya instead of Darghān, this being indicated in parentheses.

In the number giving the distance in farsakhs from Jurjāniya to Āmūya the editor reads a qāf (=100) giving 105 (at 256:4). The MS is as easily read with one dot instead of two as a fa' (=80), giving an 85, which is consistent with the rest of the passage. Hence, dropping the five (presumably for turns in the road).

$AH = 80 \text{ farsakhs} = 80 \times 3 \text{ miles} = 240 / 56\frac{2}{3} \text{ degrees}$

(256:6) $= 4;14,7^\circ$,

and $TN = \sin AH = 4;25,52.$

From 254:5, $\sin GM = 41;5,22$, so, using the expression derived in Section 86 above for 254:14,

$NF = TN \cdot (\sin GM)/R$

$= 182;4,9,57,20/R = 3;2,4.$

By the Pythagorean theorem

$$\text{Sin}\,AC = TF = \sqrt{TN^2 - NF^2}$$
$$= \sqrt{19;38,5,5,4 - 9;12,28,10,16}$$
(256:10) $$= 3;13,44,39,$$

so the value in the text is truncated, not rounded.

$$\text{So } \widehat{AC} = \text{arc Sin } 3;13,44 = 3;5,6^\circ, \text{ and}$$
$$\varphi_H = \widehat{EC} = \widehat{AC} + \widehat{EA} = \widehat{AC} + \bar{\varphi}_A$$
$$= 3;5,6^\circ + 47;43^\circ \qquad \text{(from 47:43)}$$
(256:12) $$= 50;48,6^\circ.$$

Now, using the expression derived above for 255:6,

$$\text{Sin}'\Delta\Lambda_{HA} = NF \cdot R/\text{Sin } \bar{\varphi}_H$$
$$= 3;2,4 \times R/46;29,52$$
(256:14) $$= 3;54,57,$$

so the result in the text is 0;0,1 short. Except for this, all computations in this passage are precise.

Finally, $\Delta\Lambda_{HA}$ = arc Sin 3;54,57 = 3;44,30°, so the longitude of Āmūya is

$$\Lambda_H = \Lambda_{\text{Jurjāniya}} + \Delta\Lambda_{HA}$$
(250:16) $$= 84;0,54^\circ + 3;44,30^\circ$$
(256:16) $$= 87;45,24^\circ.$$

88. The Azimuth Difference of Āmūya and Bukhārā from Darghān (257:1–17).

The passage from 257:1 to 259:18 is likewise a digression from the main traverse, undertaken to calculate the coordinates of Bukhārā from those of Darghān and Āmūya. The method is essentially the complicated technique explained in Sections 80–84. It is the only example of really poor computation in the entire Taḥdīd, involving, as will be seen, two egregious blunders and three less serious errors. The result gives a badly erroneous position for Bukhārā, but it does not affect the main objective.

The first step is an application of the trapezoid algorism to find the angle subtended at Darghān by Āmūya and Bukhārā. The horizon of Darghān is substituted for the equator as the fundamental circle. Abū Rayḥān refers to Figure 63. In order to make this easier we have relabelled the latter (and Figure C63) as the separate drawing Figure C63.1.

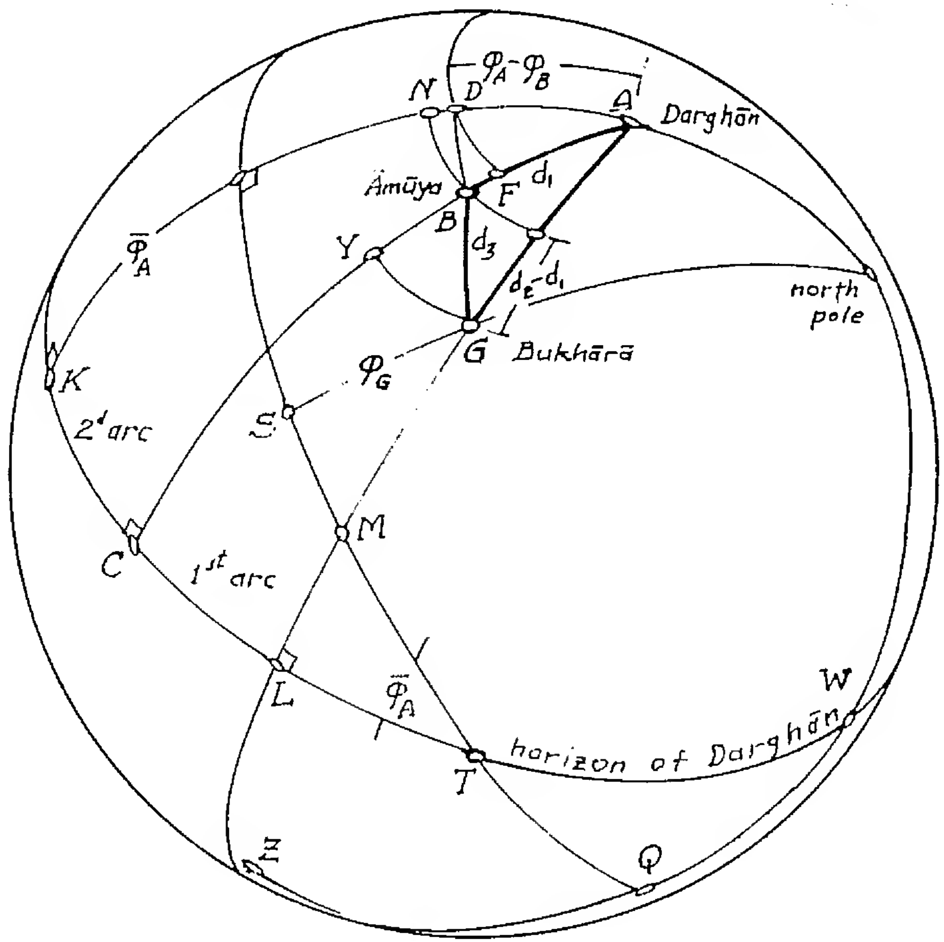


Figure C63.1

The "first displacement", $d_1 = AB$, is 35 farsakhs. Subtracting a tenth of this to straighten out the road, we have $35 - 3.5 = 31.5$ which the text calls 31. This is fair enough, but the conversion into miles is $31 \times 3 = 93$, not 63! The latter is intended, for the conversion into degrees is

(257:5) $\quad 63 \text{ miles} = 63 \,/\, 56\tfrac{2}{3} \text{ degrees} = 1;6,42^\circ.$

In like manner the "second displacement", $d_2 = AG$, is 36 farsakhs. Again reducing by a tenth, gives $36 - 3.6 = 32.4$, rounded to 32. There is a second gross error in the conversion to miles. $32 \times 3 = 96$, but Bīrūnī again writes a six for a nine. In degrees it is

$$66 \text{ miles} = 66 \,/\, 56\tfrac{2}{3} \text{ degrees} = 1,9,53^\circ.$$

The "third displacement" is

(257:8) $\quad d_3 = BG = (20-2) \text{ farsakhs} = 18 \times 3 \text{ miles}$

$$= 54 \quad 56\tfrac{2}{3} \text{ degrees} = 0;57,11^\circ.$$

Taking sines

(257:14) $\quad d_1: \quad \text{Sin } AB = \text{Sin } 1;6,42^\circ = 1;9,51.$

$$d_2: \quad \text{Sin } AG = \text{Sin } 1;9,53^\circ = 1;13,10.$$

The trapezoid for the algorism in Section 77 is BOGY in which the side

(257:10) $\quad OG = d_2 - d_1 = 0;3,11^\circ.$

The computation is

$$\angle A = \widehat{LC} = \text{arc Crd}\left(\sqrt{(\text{Crd}^2 d_3 - \text{Crd}^2 OG)} \cdot \text{Sin } d_2/\text{Sin } d_1 \cdot R/\text{Sin } d_2\right)$$

$$= \text{arc Crd}\left(\sqrt{(0;59,53^2 - 0;3,20^2)} \times 1;13,10/1;9,51 \times 60/1;13,10\right)$$

(257:15) $\quad = \text{arc Crd } \left(1;4,9,55,14 \times 60/1;13,10\right)$

$$= \text{arc Crd } 50;50,57$$

(257:17) $\quad = 50;8,33^\circ.$

Bukhārā and Samarqand were the two principal cities of Sughd, the ancient Sogdiana, the region north of the Oxus and southeast of the Aral Sea (LeStr., p. 460).

89. The Azimuth of Āmūya from Darghān (258:1-12)

Now the same algorism is applied with the same fundamental circle, the horizon of Jurjāniya, to the trapezoid NDFB in order to calculate the arc KC.

To obtain the chord of the diagonal, DB, we use the fact that its arc is a parallel of latitude and in degrees equal to

(258:1) $\quad \Lambda_B - \Lambda_A = 87;45,24^\circ - 86;23,56^\circ$

(256:16, 255:7) $\quad = 1;21,28^\circ.$

So

$$DB = \text{Crd } \Delta\Lambda \cdot (\text{Cos } \varphi_B)/R$$

(240:3) $\quad = \text{Crd } 1;21,28^\circ \times \text{Cos } 42;17^\circ/R$

(256:12) $\quad = 1;25,11 \times 46;29,52 \,/\, 60$

(258:2) $\quad = 66;0,50,8,32 \,/\, 60$

$$= 1;6,1.$$

There is an error in the last digit of $\text{Crd } \Delta\Lambda$, which should be 19, not 11. Multiplication of this chord by $\text{Cos } \varphi_B$ is to effect a proper reduction of scale, since its arc is in a parallel, not a great circle Next

$$\widehat{BF} = (\varphi_A - \varphi_B) - d_1 = DA - d_1$$

(255:1, 256:12) $\quad = (40;30,17^\circ - 39;11,54^\circ) - 1;6,42^\circ$

(257:5, 258:5) $\quad = 0;11,41^\circ.$

The main computation is

$$KC = \text{arc Crd}\left(\sqrt{\text{Crd}^2 DB - \text{Crd}^2 BF}\right) \cdot \text{Sin } DA/\text{Sin } d_1 \cdot R/\text{Sin } DA$$

(257:14) $\quad = \text{arc Crd}\left(\sqrt{(1;6,1^2 - 0;12,11^2)} \cdot \text{Sin } 1;18,23/1;9,51 \times 60/\text{Sin } 1;18,23\right)$

$$= \text{arc Crd}\left(\sqrt{(1;12,38,12,1 - 0;2,28,26,1)} \cdot 1;22,2/1;9,5 \times 60/1;22,2\right)$$

(258:10) $\quad = \text{arc Crd } (1;10,19 \times 60 \,/\, 1;22,2)$

$$= \text{arc Crd } 51;25,49$$

(258:12) $\quad = 50;45,21^\circ.$

There are two errors; the third digit of Crd BF should be 14, not 11, and that of Crd DA should be 5, not 2.

90. Calculation of the Latitude of Bukhārā (258:13 – 259:4)

Application of the sine law to the right triangle TQW, as in Section 83 above, gives

$$\text{Sin}\,(WQ = \angle \bar{M})/\text{Sin}\,(TW = KL) \;=\; \text{Sin}\,(\angle T = \bar{\varphi}_A)/\text{Sin}\,(Q = 90^\circ).$$

Hence

$$\angle M = \text{arc Cos}\,(\text{Sin}\,KL \cdot \text{Cos}\,\varphi_A/R)$$
$$= \text{arc Cos}\,(\text{Sin}\,79;6,6^\circ \cdot \text{Cos}\,40;30,17^\circ / R)$$
(258:14, 255:1) $$= \text{arc Cos}\,(59;55,12 \times 45;37,16 / 60)$$
$$= \text{arc Cos}\,(273;29,24,54,32 / 60)$$
(258:17) $$= \text{arc Cos}\,45;33,29$$
$$= 40;35,59^\circ.$$

This computation contains three errors. The value given for Sin KL is badly off. It should be 58;55,4. Accepting the text's erroneous value, its product with Cos φ_A is obtained by us as 2733;37,1,7,12 rather than the value in the text. And arc Sin 45;33,29 is 49;24,19°, not 49;24,1 as the author has it.

Now, applying the sine law to triangle LMT gives

$$\text{Sin}\,LM/\text{Sin}\,(\angle T = \bar{\varphi}_A) \;=\; \text{Sin}\,LT/\text{Sin}\,\angle M,$$

whence

$$LM = \text{arc Sin}\,(\text{Sin}\,LT \cdot \text{Cos}\,\varphi_A/\text{Sin}\,\angle M)$$
(258:15) $$= \text{arc Sin}\,(\text{Sin}\,10;53,54^\circ \cdot \text{Cos}\,40;30,17/\text{Sin}\,40;35,59^\circ)$$
(258:18) $$= \text{arc Sin}\,(11;20,39 \times 45;37,16 / 39;2,46)$$
$$= \text{arc Sin}\,(517;32,0,33,24/39;2,46)$$
(258:20) $$= \text{arc Sin}\,(13;15,19)$$
$$= 12;45,47^\circ.$$

Here, for Sin LT we have restored the second digit as a 20 from the 24 of the printed text, the former evidently having been in

the original. The final digit should have been a 38, rather than the 39 which appears. For Cos φ_A we have here used a terminal digit of 16 in order to produce the result of the text. See the remark in the last paragraph of Section 86. The quotient which equals Sin LM is in fact 13;15,16, not 13;15,19 as in the text.

Next, calculate

$$MG = LG - LM = \bar{d}_2 - LM$$
(259:1) $$= 88;50,7^\circ - 12;45,47^\circ = 76;4,20^\circ.$$

Finally, application of the law of sines to triangle MGS gives

$$\text{Sin}\,\varphi_G/\text{Sin}\,\angle M \;=\; \text{Sin}\,MG/\text{Sin}\,(\angle S = 90^\circ)$$

so that

$$\varphi_G = \text{arc Sin}\,(\text{Sin}\,MG \cdot \text{Sin}\,\angle M/R)$$
(258:18) $$= \text{arc Sin}\,(58;14,9 \times 39;2,46 / 60)$$
$$= \text{arc Sin}\,(2273;52,58,8,54/60)$$
(259:4) $$= \text{arc Sin}\,37;53,53$$
$$= 39;10,15^\circ,$$

for the latitude of Bukhārā. The actual latitude of this city is 39;47°, but the result above is much closer to Bīrūnī's received value of 39;20° given at 259:16.

91. Calculation of the Longitude of Bukhārā (259:5 – 18)

This is one more application of the standard algorism of Section 77 although the trapezoid itself does not appear on Figure C63.1. We have

$$\Delta\lambda_{AS} = \text{arc Crd}\left(\sqrt{(\text{Crd}^2 AG - \text{Crd}^2 \Delta\varphi)\text{Cos}\,\varphi_A/\text{Cos}\,\varphi_G \cdot R/\text{Cos}\,\varphi_A}\right)$$
(257:7, 255:1) $$= \text{arc Crd}\left(\sqrt{(\text{Crd}^2 1;9,53^\circ - \text{Crd}^2 1;20,2^\circ)\text{Cos}\,40;30,17^\circ/\text{Cos}\,39;10,15^\circ \cdot 60/\text{Cos}\,40;30,17^\circ}\right)$$
(259:7, 255:3) $$= \text{arc Crd}\left(\sqrt{(1;13,10^2 - 1;23,49^2) \times 45;37,16/46;30,57 \times 60/45;37,16}\right)$$

$$= \text{arc Crd}\left(\sqrt{0;27,51,52,21 \times 45;37,16/46;30,57 \times 60/45;37,16}\right)$$

(259:8) $$= \text{arc Crd}\left(\sqrt{0;27,19,42,52} \times 60/45;37,16\right)$$

$$= \text{arc Crd } 0;53,15 = 0;50,[5]1^{\circ},$$

where

$$\Delta\varphi = \varphi_A - \varphi_B = 40;30,17^{\circ} - 39;10,15^{\circ}$$
$$= 1;20,2^{\circ}.$$

The entire computation is accurate except that Crd AG and the square root have both been truncated rather than rounded. Also the terminal digit of the result must be restored to the 51 of the original from the 11 of the translation and printed text.

Hence the longitude of Bukhārā is

$$\Lambda_G = \Lambda_A + \Delta\Lambda$$

(255:7) $$= 86;23,56^{\circ} + 0;50,[5]1^{\circ}$$

(259:14) $$= 87;14,47^{\circ}$$

This, as it turns out, is close to the traditional $87;30^{\circ}$.

92. The Distance Between Balkh and Bukhārā Calculated (260:1 – 261:2)

In this section Abū Rayḥān reverses his usual procedure and uses the coordinates of two localities to determine the great circle distance between them, thus to check the road distances reported by caravans. The longitude difference between Balkh and Bukhārā is

(252:5) $$\Delta\Lambda = \Lambda_2 - \Lambda_1 = 90;19,48^{\circ} - 87;30^{\circ}$$

(260:3) $$= 2;49,48^{\circ}.$$

The latitude difference is

(251:3) $$\Delta\varphi = \varphi_1 - \varphi_2 = 39;20^{\circ} - 36;41,36^{\circ}$$

(260:8) $$= 2;38,24^{\circ}$$

The expression derived in Section 74 above can be written as

$$AB = \text{arc Crd}\sqrt{(\text{Crd }\Delta\Lambda \cdot \text{Cos }\varphi_1/R)\cdot(\text{Crd }\Delta\Lambda \cdot \text{Cos }\varphi_2/R) + \text{Crd}^2\,\Delta\varphi}\,.$$

Substituting,

(251:13) $$AB = \text{arc Crd}\sqrt{(2;57,5[2] \times 46;24,30/R)\cdot(2;57,5[2] \times 48;6,38/R) + 2;45,52^2}$$

(260:6) $$= \text{arc Crd}\sqrt{(137;34,29,44,0/R)\cdot(142;37,15,5[6],56/R) + 2;45,52^2}$$

(260:9) $$= \text{arc Crd}\sqrt{(2;17,34 \times 2;22,37) + 7;38,31,45,4}$$

The accurate value of Crd $\Delta\Lambda$ is 2;57,48. Hence the amount given in the MS and the printed text and translation, 2;57,55, is wrong. However, this is not what was used by Bīrūnī, for only 2;57,52, as shown restored above, checks with the computations following it in the text. A second scribal error has been restored, from 50 to 56, in the expression above headed by (260:6).

(260:10) $$AB = \text{arc Crd}\sqrt{13;5,31,3,2}$$

$$= \text{arc Crd } 3;36,56 = 3;27,11^{\circ}$$

(260:12) $$= 3;27,11^{\circ} \times 56;40 \text{ miles/degree} = 195;40,23 \text{ miles}$$

$$= 65;13,28 \text{ farsakhs}.$$

There is an error in the extraction of the root; it should be 3;37,6. The reported road distance is about five farsakhs more. If, say, a tenth of the great circle distance is added for turns in the road, the two results are indeed quite close.

93. The Coordinates of Nīshāpūr (261:3 – 263:18)

The element of uncertainty in fixing upon a longitude for Balkh leads Bīrūnī to comment upon the complications of such determinations in general, and in particular to a discussion of the various results obtainable for Nīshāpūr (cf. Section 11). In the India (transl., vol. I, p. 305), written after the Taḥdīd, he mentions his intention of writing a book on the longitude of Nīshāpūr. A search through

Boilot has revealed no trace of this work, although it may be one of a number of non-extant geographical treatises listed in Bīrūnī's bibliography.

Much of the material in the latter part of this section is clearly based on extensive computation, not shown here. He may well have been already at work on the projected Nīshāpūr book, of which preliminary results were inserted into the Taḥdīd at this point.

Since the accurately determined latitude of the place is $36;13^\circ$, Manṣūr b. Ṭalḥa's value of $36;10^\circ$ (261:4) is quite good, and at least in this instance he is undeserving of Abū Rayḥān's denigration.

There follows a series of reports of longitude differences from other sources. They are tabulated below with the corresponding accurate value set opposite

		Reported	Accurate
(1) Nīshāpūr–Baghdād, by Ibn Hamdūn and Manṣūr	(261:5)	$12;30^\circ$	$14;23^\circ$
(2) Nīshāpūr–Samarra, by the Banū Mūsā b. Shākir	(261:9)	10°	$14;57^\circ$
(3) Nīshāpūr–Mecca	(272:7)	$20;30^\circ$	$19;0^\circ$
(4) Nīshāpūr–Balkh	(262:7)	10°	$8;16^\circ$
(5) Mecca–Baghdād, by Habash	(262:11)	3°	$4;37^\circ$

Concerning Manṣūr, the Banū Mūsā, al-Makkī, Ḥabash, and the books mentioned in this passage, see Sections 23, 15, 37, and 68 above.

Considering the means available to medieval astronomers for measuring longitude differences the inaccuracy of the determinations listed is not surprising. The 8° at 262:9 for the $\Delta\Lambda$ between Mecca and Baghdād is obtained by subtracting (1) from (3). As Bīrūnī remarks, it is badly out, and Ḥabash's 3° is much better.

Neither Marv al-Rūd nor Baghshūr presently exist, at least not under these names. The site of the former is, roughly speaking, midway on the meridian between Marv and Herāt. Baghshūr was probably a short distance west of Marv al-Rūd (LeStr., pp. 397-400, 413).

The remainder of the passage (263:1 – 18) we take to be a report of computations carried out by Bīrūnī himself elsewhere. No attempt has been made here to recompute the results.

An application of the trapezoid algorism presumably gave the $\Delta\Lambda$ of $7;18,12^\circ$ for Rayy–Nīshāpūr. Thence

(263:4) $$\Lambda_{\text{Nīshāpūr}} = \Lambda_{\text{Rayy}} + \Delta\Lambda = 85^\circ + 7;18,12^\circ$$
$$= 92;18,12^\circ,$$

a result so far removed from the other values that hence, perhaps, it is not even reported in the text.

The algorism may have been applied a second time to the arc Balkh–Nīshāpūr to give a $\Delta\Lambda$ of $4;33,32^\circ$, whence

$$\Lambda_{\text{Nīshāpūr}} = \Lambda_{\text{Balkh}} - \Delta\Lambda$$
(261:1) $$= 91^\circ - 4;33,32^\circ$$
$$= 86;26,28^\circ.$$

Next, an application of the complicated technique illustrated in Figure C63 to the triangle Nīshāpūr–Jurjān–Jurjāniya may have produced the $\Delta\Lambda$ for Jurjān–Nīshāpūr of $4;31,56^\circ$. From it

$$\Lambda_{\text{Nīshāpūr}} = \Lambda_{\text{Jurjān}} + \Delta\Lambda$$
(243:18) $$= 80;14,1^\circ + 4;31,56^\circ$$
(263:12) $$= 84;45,57^\circ.$$

The same technique applied to the triangle Nīshāpūr–Jurjāniya–Balkh may have been the means of obtaining, for Jurjāniya–Nīshāpūr, $\Delta\Lambda = 1;56,58^\circ$, and

$$\Lambda_{\text{Nīshāpūr}} = \Lambda_{\text{Jurjāniya}} + \Delta\Lambda$$
(250:16) $$= 84;0,54^\circ + 1;56,58^\circ$$
$$= 85;57,52^\circ.$$

CHAPTER VIII. THE SOUTHERN TRAVERSE

94. The Longitude Difference Between Baghdād and Shīrāz (263:19 – 264:11)

There now remains only the final leg, Balkh-Ghazna, of the original traverse. Nevertheless, Bīrūnī chose to postpone its computation, and commences an independent run. The initial leg is from Baghdād to Shīrāz,

$$\widehat{AB} \;=\; (170 \text{ farsakhs})\frac{9}{10} \;=\; 153 \times 3 \text{ miles}$$

(264:11) $\quad = \; 459 \text{ miles} \;/\; 56\frac{2}{3} \text{ miles/degree} \;=\; 8;6,0^\circ.$

For Shīrāz, Ṣufī's latitude $\varphi_1 = 29;36^\circ$ is only two minutes low. The latitude of Baghdād, $\varphi_2 = 33;25^\circ$, from 100:9, say, so

(264:4) $\quad \Delta\varphi = \varphi_2 - \varphi_1 \;=\; 33;25^\circ - 29;36^\circ \;=\; 3;49^\circ.$

Then (cf. Section 77)

$$\Delta\Lambda = \text{arc Crd}\left(\sqrt{(\text{Crd}^2 AB - \text{Crd}^2\Delta\varphi)\text{Cos }\varphi_2/\text{Cos }\varphi_1}\cdot R/\text{Cos }\varphi_2\right)$$

(238:8) $\quad = \text{arc Crd}\left(\sqrt{(8;28,32^2-3;59,46^2)\times 50;4,52/52;10,10}\right.$ $\left.\times\, 60/50;4,52\right)$

$$= \text{arc Crd}\sqrt{55;51,58,5,48\times 50;4,52/52;10,10}\times 60/50;4,52)$$

$$= \text{arc Crd}\sqrt{2797;50,17,44,44,13,36/\text{Cos }\varphi_1}\cdot R/\text{Cos }\varphi_2).$$

Up to this point everything is fine, but for the next step Bīrūnī divides by Cos φ_2 instead of Cos. φ_1. That this has indeed happened can easily be verified from the text, for except for the last digit (a misprint in the translation) the result in 264:7 is identical with the number under the last radical above. That is, division by Cos φ_2 cancels the previous multiplication by Cos φ_2. Hence the result is badly out.

The erroneous result is

$$\Delta\Lambda = \text{arc Crd } 8;57,16 \;=\; 8;33,32^\circ.$$

Aside from the blunder, two results are truncated rather than rounded, and a third is one unit low in the last digit – all other results are precise to three sexagesimal places.

Finally,

(239:6) $\quad \Lambda_{\text{Shīrāz}} = \Lambda_{\text{Baghdād}} + \Delta\Lambda = 70^\circ + 8;33,32^\circ$

(264:1) $\quad = 78;33,32^\circ$

95. The Longitude Difference Between Shīrāz and Zaranj (264:12 – 266:5)

The second leg is another big jump, in length along a great circle

$$\widehat{AB} \;=\; (78+47+70)\frac{6}{7} \text{ farsakhs} \approx 168 \times 3 \text{ miles}$$

(265:6) $\quad = 504 \times 56\frac{2}{3} \;=\; 8;53,39^\circ.$

The latitude adopted for Zaranj, $\varphi_2 \doteq 30;52^\circ$ from 265:1, is eight minutes low if Zaranj is the site of modern Zābol. From 264:3 the latitude of Shīrāz is $\varphi_1 = 29;36^\circ$. Hence, applying the algorism of Section 77,

(265:7) $\quad \Delta\varphi = \varphi_2 - \varphi_1 = 1;16,0^\circ,$

and

$$\Delta\Lambda = \text{arc Crd}\left(\sqrt{(\text{Crd}^2\, 8;53,39^\circ - \text{Crd}^2\, 1;16,0^\circ)\text{ Cos }30;52/\text{Cos }29;36^\circ}\cdot R/\text{Cos }30;52^\circ\right)$$

(265:10, 264:6) $\quad = \text{arc Crd}\left(\sqrt{(9;18,16^2-1;19,35^2)\times 51;30,6/52;10,10}\;\; 60/51;30,6\right)$

$$= \text{arc Crd}\left(\sqrt{84;48,48,9,51\times 51;30,6/52;10,10}\times 60/51;30,6\right)$$

(265:13) $\quad = \text{arc Crd }(9;9,1\times 60\,/\,51;30,6)$

$$= \text{arc Crd }10;39,37 \;=\; 10;11,36^\circ$$

So

$$\Lambda_{\text{Zaranj}} = \Lambda_{\text{Shīrāz}} + \Delta\Lambda$$

$$(264:11) \quad = 78;33,32^\circ + 10;11,36^\circ$$

$$(265:15) \quad = 88;45,8^\circ \approx 89^\circ.$$

Except for two cases where a result has been truncated rather than rounded all these computations are precise to three sexagesimal digits.

Another $\Delta\Lambda$ is given (266:1), for Zaranj-Nīshāpūr, with no computations. We conjecture this to be another excerpt from Bīrūnī's separate investigation of the longitude of Nīshāpūr (see Section 93 above). From this,

$$\Lambda_{\text{Nīshāpūr}} = \Lambda_{\text{Zaranj}} - \Delta\Lambda$$

$$(266:3) \quad = 89^\circ - 4;12,16^\circ$$

$$= 84;47,44^\circ,$$

where it is preferable to read a 47 in the MS text for the second digit of the number above, instead of 46 as in the printed text and the translation.

The metropolis Zaranj, now ruined, was the capital of Sijistan (modern Sīstān) a once-populous region in the southwest corner of modern Afghanistan (LeStr., pp. 334-8). Sīstān was the seat of the legendary first Iranian dynasty, and the home of the epic hero, Rustām. Nīmrūz (265:16) is Persian for "mid-day". Sīstān was thus called in relation to Khurāsān, probably to indicate that the one lay on the same meridian as the other.

Quhistān, from Persian Kūhistān for "mountain-land", with its Arabic equivalent Jibāl, were medieval names for the central province of Īrān, the ancient Media (LeStr., p. 185).

We find no other mention in the medieval literature of the Zaranj astronomer Abū al-Ḥasan Aḥmad b. Muḥammad b. Sulaiman and his eleven-meter quadrant (264:15).

96. The Longitude Difference Between Balkh and Ghazna (266:6 - 267:9)

Bīrūnī now reverts to the northern traverse and fills in the last leg. For φ_2, the latitude of Ghazna, he reduces his own observations of meridian solar altitude made at the summer and winter solstices of the year 1012, with a quadrant of a circle of radius about $3\frac{1}{4}$ meters:

$$\varepsilon = \frac{\max\varphi - \min\varphi}{2}$$

$$= \frac{1}{2}(80;0^\circ - 32;50^\circ)$$

$$(266:10) \quad = 23;35^\circ.$$

Hence

$$\varphi_2 = 90^\circ - (\min\varphi + \varepsilon) = 90^\circ - (32;50^\circ + 23;35^\circ)$$

$$(266:11) \quad = 33;35^\circ,$$

a determination two minutes high.

Since, from 251:5, the latitude of Balkh is $36;41,36^\circ = \varphi_1$,

$$(266:12) \quad \Delta\varphi = \varphi_1 - \varphi_2 = 3;6,36^\circ.$$

The great circle distance is

$$\widehat{AB} = 80 \times \frac{4}{5} \text{ farsakhs}$$

$$= 64 \times 3 \text{ miles}$$

$$= 192 / 56\frac{2}{3} \text{ degrees}$$

$$(266:14) \quad = 3;23,18^\circ$$

So (cf. Section 22),

$$\Delta\Lambda = \text{arc Crd}\left(\sqrt{(\text{Crd}^2 3;23,18^\circ - \text{Crd}^2 3;6,36^\circ)\text{ Cos } 33;35^\circ/\text{Cos } 36;41,36^\circ} \cdot R/\text{Cos } 33;35^\circ\right)$$

$$(251:13) \quad = \text{arc Crd}\left(\sqrt{(3;32,52^2 - 3;15,23^2)\times 49;59,5/48;6,38\times 60/49;59\cdot 5}\right)$$

$$= \text{arc Crd}(1;26,4 \times 60/49;59,5)$$

$$(267:2) \quad = \text{arc Crd } 1;43,21$$

$$= 1;38,42^\circ.$$

The accurate value of the square root above is 1;26,6,50, and the last digit of the final chord should be 19, not 21. Otherwise the computation is accurate in the text.

The result is

$$\Lambda_{\text{Ghazna}} = \Lambda_{\text{Balkh}} + \Delta\Lambda$$

(261:1) $$= 91^\circ + 1;38,42^\circ$$

$$= 92;38,42^\circ \approx 93^\circ.$$

For Tukhāristān and Zābulistān, see Section 41. Rakhaj (rather, Rukhkhaj, LeStr., p. 339), in Sīstān, is the valley of the Qandahār (Kandahar) River.

97. The Longitude Difference Between Zaranj and Bust (267:10 – 269:10)

The southern traverse is now resumed, with the endpoint of a leg taken at Bust, the modern Qalʿa Bīst. For its latitude, φ_1, Bīrūnī makes use of an old zīj owned by one ʿAlī b. Muḥammad al-Wishjardī, otherwise unknown to the literature. Ptolemy's Tetrabiblos (268:13) is to astrology what his Almagest is to astronomy. The epoch of the zīj was 29 August 284, the Era of the Martyrs, hence its tables were probably based on the Coptic calendar. The latter, used by the Christians of Egypt, was known to the astronomers of medieval Islam although it was not among the most common calendars. The zīj must have been very early, since notes at the end reported eclipse observations between 708/9 and 718/9 A.D.

The zīj apparently contained a statement that the meridian solar altitude at Bust on the winter solstice was 34;10°. Hence, assuming that it used Ptolemy's value for the inclination of the ecliptic (Section 19)

$$\bar{\varphi}_1 = \min \varphi + \varepsilon = 34;10^\circ + 23;51,20^\circ = 58;1,20^\circ, \text{ and}$$

(268:5) $$\varphi_1 = 31;58,40 \approx 32^\circ.$$

Taking Bīrūnī's ε,

(268:9) $$\bar{\varphi}_1 = 34;10^\circ + 23;35^\circ = 57;45^\circ$$

$$\varphi_1 = 32;15^\circ$$

If our identification of the locality is correct, with its latitude of 31;28°, then the traditional value he reports in 267:11 as used by the local inhabitants, 31;10° is much better than the one he adopts. Taking for Zaranj $\varphi_2 = 30;52^\circ$ from 265:1

(269:2) $$\Delta\varphi = 32;15^\circ - 30;52^\circ = 1;23^\circ.$$

Also $$\widehat{AB} = 60 \times \frac{5}{6} \text{ farsakhs}$$

$$= 50 \times 3 \text{ miles}$$

$$= 150 \,/\, 56\tfrac{2}{3} \text{ degrees}$$

(269:4) $$= 2;38,49^\circ$$

Hence, applying the familiar expression in Section 77,

$$\Delta\Lambda = \text{arc Crd}\left(\sqrt{\text{Crd}^2 2;38,49^\circ - \text{Crd}^2 1;23^\circ) \cdot \text{Cos } 30;52^\circ/\text{Cos } 32;15^\circ} \cdot \text{R/Cos } 30;52^\circ\right)$$

(256:8) $$= \text{arc Crd}\left(\sqrt{(2;46,19^2 - 1;26,55^2) \times 51;30,6/50;44,37} \times 60/51;30,6\right)$$

$$= \text{arc Crd}\left(\sqrt{5;35,6,43,36 \times 51;30,6/50;44,37} \times 60/51;30,6\right)$$

(269:8) $$= \text{arc Crd } (2;22,51 \times 60/51;30,6)$$

$$= \text{arc Crd } 2;46,25 = 2;37,30^\circ$$

There are two computational slips. Crd AB is in fact 2;46,17,52, hence badly rounded, and, which is more serious, the arc chord at the end should be 2;38,55°.

Accepting Bīrūnī's result,

$$\Lambda_{\text{Bust}} = \Lambda_{\text{Zaranj}} + \Delta\Lambda$$

(265:15) $$= 89^\circ + 2;37,30^\circ$$

(269:10) $$= 91;37,30^\circ.$$

98. The Longitude Difference Between Bust and Ghazna (265:11 – 266:5)

This is the terminal leg of the southern traverse. Since, from 266:11 the latitude of Ghazna is $_1 = 33;35^{\circ}$, and from 268:9 that of Bust is $\varphi_2 = 32;15^{\circ}$,

(269:13) $\Delta\varphi = \varphi_1 - \varphi_2 = 1;20^{\circ}$,

$$\widehat{AB} = 80 \times \frac{5}{6} \text{ farsakhs} = 66\frac{2}{3} \text{ farsakhs}.$$

Dropping the fraction,

$$\widehat{AB} = 66 \times 3 \text{ miles}$$

$$= 198 \,/\, 56\tfrac{2}{3} \text{ degrees}$$

(269:15) $= 3;29,39^{\circ}$,

from the second digit of which Abū Rayḥān has inadvertently dropped the 2. Applying the trapezoid algorism of Section 77,

$$= \text{arc Crd}\left(\sqrt{(\text{Crd}^2 3;9,39^{\circ} - \text{Crd}^2 1;20^{\circ})\text{Cos } 32;15^{\circ}/\text{Cos } 33;35^{\circ}} \cdot \text{R/Cos } 32;15^{\circ}\right)$$

(269:7, 266:16) $= \text{arc Crd}\left(\sqrt{(3;18,38^2 - 1;23,46^2) \times 50;44,37/49;59,5} \times 60/50;44,37\right)$

$= \text{arc Crd } (3;1,28 \times 60/50;44,37)$

(270:3) $= \text{arc Crd } 3;34,34$

$= 3;24,56^{\circ}$

In this computation Crd $3;9,39^{\circ}$ should have been rounded off to 35 in the last digit, not 38, and the two restorations in the translation are correct.

The result is

$$\Lambda_{\text{Ghazna}} = \Lambda_{\text{Bust}} + \Delta\Lambda$$

(269:10) $= 91;37,30^{\circ} + 3;24,56^{\circ}$

(270:4) $= 95;2,26^{\circ}$,

a second result for the final objective.

99. The Longitude Difference Between Zaranj and Ghazna (270:6 – 271:12)

At this point Bīrūnī decides to eliminate one leg by making a single computation for the long arc from Zaranj direct to Ghazna. His attitude towards the distance seems somewhat cavalier, for if the separate distances Zaranj–Bust–Ghazna are added we obtain $80 + 60 = 140$ farsakhs, whereas he takes 120 for the total in 270:9. Of course, the three points presumably are not on a great circle, so the total should be somewhat less than the sum. In any event,

$$\widehat{AB} = 120 \times \frac{5}{6} \text{ farsakhs}$$

$$= 100 \times 3 \text{ miles}$$

$$= 300 \,/\, 56\tfrac{2}{3} \text{ degrees}$$

(270:10) $= 5;17,39^{\circ}$.

For the latitudes, that of Ghazna is $\varphi_1 = 33;35^{\circ}$ from 266:11, and that of Zaranj is $\varphi_2 = 30;52^{\circ}$ from 265:1. So

(270:8) $\Delta\varphi = \varphi_1 - \varphi_2 = 2;43^{\circ}$.

Applying the expression of Section 77,

$$\Delta\Lambda = \text{arc Crd}\left(\sqrt{(\text{Crd}^2 5;17,39^{\circ} - \text{Crd}^2 2;43^{\circ})\text{Cos } \varphi_2/\text{Cos } \varphi_1} \cdot \text{R/Cos } \varphi_2\right)$$

(270:8) $= \text{arc Crd}\left(\sqrt{(5;32,32^2 - 2;50,41^2)\text{Cos } 30;52^{\circ}/\text{Cos } 33;35^{\circ}} \times 60/\text{Cos } 30;52^{\circ}\right)$

(265:10, 266:16) $= \text{arc Crd}\left(\sqrt{22;37,25,37,3 \times 51;30,6/49;59,5} \times 60/51;30,6\right)$

(270:14) $= \text{arc Crd } (4;49,41 \times 60 \,/\, 51;30,6)$

(271:1) $= \text{arc Crd } 5;37,29 = 5;22,24^{\circ}$.

The only computational slip in this determination is in Crd AB, the last digit of which rounds to 31 not 32 as in the text.

Hence

$$\Lambda_{\text{Ghazna}} = \Lambda_{\text{Zaranj}} + \Delta\Lambda$$

(265:15) $= 89^{\circ} + 5;22,24^{\circ}$

(271:2) $= 94;22,24^{\circ}$,

a third result for the longitude of Ghazna. The other two are

(261:1)	93°,	via Balkh, and
(270:4)	$95;2,26^{\circ}$.	via Bust.

Abū Rayḥān chooses the first of these, partly, he says, because it is near the mean of the other two. But we may well suspect him of having fudged the distance Zaranj-Ghazna in order to obtain a result which instinct told him was reasonable.

100. The Coordinates of Bust Calculated from those of Ghazna and Zaranj (271:13 – 272:16)

Having attained his final objective, Bīrūnī now turns back to redetermine the location of Bust in terms of the points on either side of it on the southern traverse. He simply reports the computations, step by step, without justification. Our Figure C64.2 supplies this, and the author must have worked from some such configuration. On it Zaranj, Bust, and Ghazna, represented by Ż, B, and G respectively are shown as being on a single great circle. This assumption is invalid, but, as will be seen below, it is implicit in the relations underlying the computation.

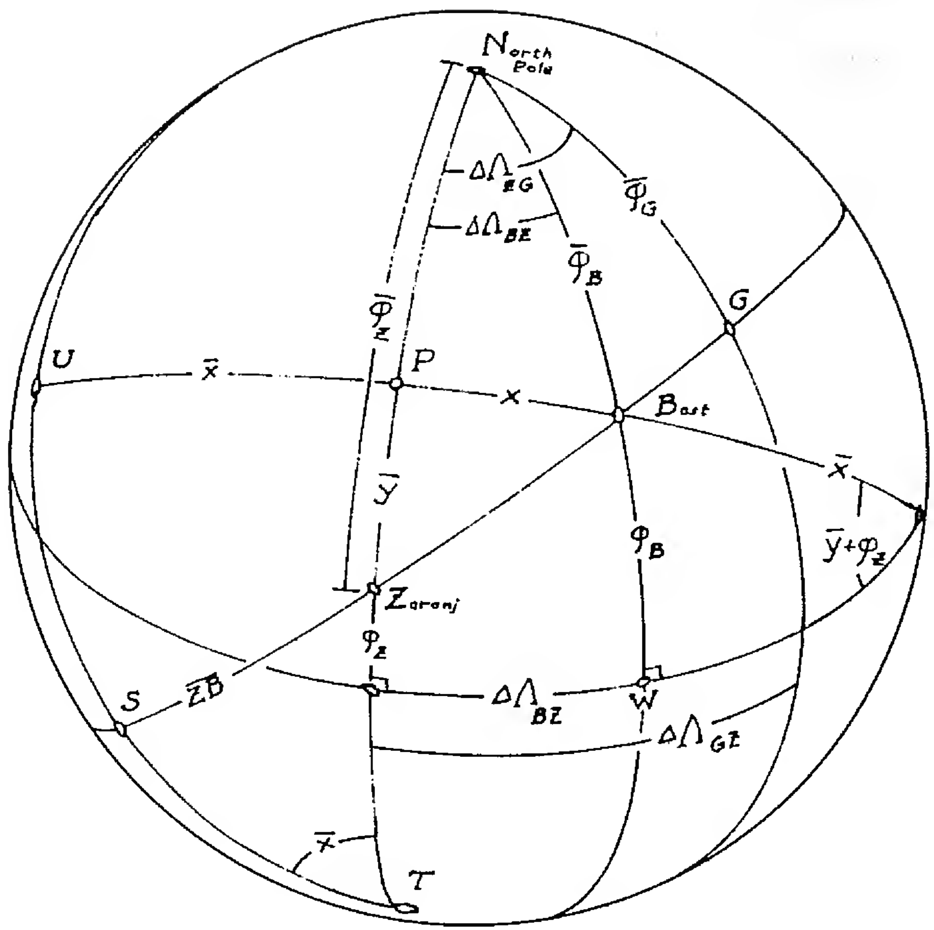


Figure C64.2

The first step is to form

(271:15, 271:1) $\left(\dfrac{\operatorname{Sin} \bar{\varphi}_G \cdot \operatorname{Sin} \Delta\Lambda_{ZG}}{\operatorname{Sin} ZG} \right) \operatorname{Sin} ZB$

(269:2) $= \left(\dfrac{\operatorname{Cos} 33;35^\circ \cdot \operatorname{Sin} 5;22,24^\circ}{\operatorname{Sin} 5;17,32^\circ} \right) \operatorname{Sin} 2;38,49^\circ$

(266:12) $= \left(\dfrac{49;59,5 \times 5;37,7}{5;32,10} \right) \times 2;46,15$

(272:1) $= 140;33,48,58,45 = n_1$,

called the "first retained number". In fact the law of sines applied to triangle NZG gives

$$\operatorname{Sin} \varphi_G / \operatorname{Sin} \angle NZG = \operatorname{Sin} ZG / \operatorname{Sin} \Delta\Lambda_{ZG}$$

from which it follows that

(A) $n_1 = \operatorname{Sin} \angle NZG \cdot \operatorname{Sin} ZB$.

Next, form

(272:2) $n_1/R = \operatorname{Sin} x = 140;34/R = 2;20,34$,

whence

$$\operatorname{arc\,Sin} x = 2;14,15^\circ$$

and

(B) $\operatorname{Sin} \bar{x} = \operatorname{Cos} x = 59;57,15 = n_2$,

the "second retained number".

These manipulations can be interpreted on the sphere, for from expression A above,

$$\operatorname{Sin} x = \frac{n_1}{R} = \frac{\operatorname{Sin} \angle NZG \cdot \operatorname{Sin} ZB}{R} ,$$

or $\operatorname{Sin} x / \operatorname{Sin} \angle NZG = \operatorname{Sin} ZB / \operatorname{Sin} 90^\circ$.

But this expression is a valid application of the law of sines to triangle PBZ formed by dropping a perpendicular from B to ZN, the meridian through Z, and $x = BP$.

Now form

(272:5) $(\operatorname{Cos} ZB) \cdot R/n_2 = \operatorname{Sin} y = (\operatorname{Sin} ZB) \cdot R/n_2$

(269:2) $= (\operatorname{Cos} 2;38,49^\circ) \cdot R/n_2 = (\operatorname{Sin} 87;21,11^\circ) \cdot R/n_2$

$= 59;56,7 \times 60 / 59;57,15$

$= 59;58,51$,

whence

(272:7) $y = \operatorname{arc\,Sin} 59;58,51 = 88;33,25^\circ$

and

$$\bar{y} = 1;26,35^\circ .$$

Interpretation of this is somewhat more involved. Apply the expression marked (B) to that marked (272:5) to obtain

$$\operatorname{Sin} y = (\operatorname{Sin} ZB) \cdot R / \operatorname{Sin} \bar{x} ,$$

or

(C) $\operatorname{Sin} y / R = \operatorname{Sin} ZB / \operatorname{Sin} \bar{x}$.

Let UST be the great circle having B as pole. Then $UP = \bar{x}$, and $SZ = \overline{ZB}$. Moreover, T will be the pole of UPB, so if we put $PZ = \bar{y}$, then $ZT = \bar{y}$, and the magnitude of the spherical angle at T will be $\bar{x}$. Then application of the law of sines to the right triangle SZT yields expression (C) above, and our interpretation of y is proved.

Next, form

(272:8) $\operatorname{Cos} (\bar{\varphi}_Z - \bar{y}) \cdot n_2 / R$

(265:1) $= \operatorname{Cos} (59;8^\circ - 1;26,35^\circ) \cdot n_2/R$

$= \operatorname{Sin} 32;18,35^\circ \times 59;57,15 / 60$

$= 32;14,11 \times 59;57,15 / 60$

(272:10) $= 32;12,42 = \operatorname{Sin} \varphi_B$.

Hence

$$\varphi_B = \operatorname{arc\,Sin} 32;12,42 = 32;28,13^\circ$$

To justify this, note that the computation assumes, substituting for the value of n_2 in expression (B) above, that

$$\text{Cos}\,(\bar{\varphi}_Z - \bar{y}) \cdot \text{Sin}\,\bar{x}/R = \text{Sin}\,\varphi_B\,,$$

or

(D) $$\text{Sin}\,\bar{x}\,/\,R = \text{Sin}\,\varphi_B/\text{Sin}\,(\overline{\bar{\varphi}_Z - \bar{y}})$$

$$= \text{Sin}\,\varphi_B/\text{Sin}\,(\bar{y} + \varphi_Z).$$

Now, V is the point where PB extended meets the equator. Hence V is the pole of NPZ, and the spherical angle at V measured by PZ $+\varphi_Z = \bar{y} + \varphi_Z$. Moreover, BV $= \bar{x}$, and application of the law of sines to triangle BWV gives expression (D) above.

Finally, form

(272:12) $$n_1/\text{Cos}\,\varphi_B = n_1/\text{Sin}\,57;31,47^\circ$$

$$= 140;33,48,58,45/50;37,13$$

$$= 2;46,37 = \text{Sin}\,\Delta\Lambda_{ZB}\,,$$

whence

(272:13) $$\Delta\Lambda_{ZB} = \text{arc Sin}\,2;46,37 = 2;39,10^\circ\,.$$

This assumes, substituting for n_1 from (A) above, that

$$\text{Sin}\,\angle\,\text{NZG} \cdot \text{Sin}\,\text{ZB}\,/\,\text{Cos}\,\varphi = \text{Sin}\,\Delta\Lambda\,,$$

or

$$\text{Sin ZB}/\text{Sin}\,\widehat{\Delta\Lambda}_{ZB} = \text{Sin}\,\bar{\varphi}_B\,/\text{Sin}\,\angle\,\text{NZG}.$$

But this follows from an application of the sine law to triangle NZB. Here, as previously, the demonstration breaks down unless Z, B, and G are on a single great circle.

There are three computational errors. Arc Sin x has been truncated; the last digit of Cos ZB should be 10, not 7; and the middle digit of Sin $(\bar{y} + \varphi_Z)$ should be 4, not 14.

The last step consists of putting

$$\Lambda_{\text{Bust}} = \Lambda_{\text{Zaranj}} + \Delta\Lambda$$

(265:15) $$= 89^\circ + 2;39,10^\circ$$

(272:14) $$= 91;39,10^\circ.$$

From 269:10 the result was 91;37,30°. Bīrūnī drops the seconds from both and takes the mean of the remainders to obtain 91;38° as the approved result.

101. The Ghazna Longitude Calculated in the Canon

Long after he had completed the Taḥdīd, Bīrūnī returned to the same problem. In the Canon, VI, 2 (pp. 609-616) he redetermined the longitude difference between Ghazna and Alexandria using essentially the same techniques, traverses, and computations. This chapter of the Canon was competently translated and commented upon in Schoy, Bestimmung, and subsequently the same material was reworked by J. H. Kramers in Comm. Vol., pp. 177-197.

The Canon determination is less elaborate than that of the Taḥdīd. Two legs: Alexandria – Raqqa and Raqqa – Baghdad, obtain the $\Delta\Lambda$ between Alexandria and Baghdad. Thence again there are two traverses, a northern and a southern. In the former the intermediate station of Balkh is omitted. In the south a single leg computes the $\Delta\Lambda$ between Shīrāz and Ghazna, omitting the stations at Zaranj and Bust. The elaborate checkings via Nīshāpūr and other substations are not mentioned.

We tabulate below the longitude differences obtained between the main stations in both books:

	Taḥdīd	Canon
Alexandria–Raqqa (295:16)	11;45,15°	11;45,15°
Raqqa–Baghdad (294:19)	6;20,43°	6;20,43°
Baghdad–Rayy (238:11)	8;5,20°	8;5,20°
Rayy–Jurjāniya (240:15)	6;1,26°	6;1,26°
Jurjāniya–Ghazna (252:3, 267:2)	7;57,36°	9;37,16°
Baghdad–Shīrāz (264:10)	8;33,32°	8;33,32°
Shīrāz–Ghazna (265:14269:10, 270:4)	16;14,2°	16;20,54°
(265:14, 271:1)	15;34,0°	

Inspection of the two columns of numbers makes it evident that the presentation in the Canon is by no means an independent determination. Abu Rayhan has taken over without change the results obtained in the Taḥdīd, recalculating only in the two instances where stations were eliminated.

Summation of the longitude differences and addition to the Baghdad longitude of 70° gives for the longitude of Ghazna

93;44,2° from the northern traverse, and
94;54,26° from the southern traverse.

Bīrūnī chooses the arithmetic mean of these as his value for the <u>Canon</u>:

$$94;19,14^\circ ,$$

it being close to his preferred result in the <u>Taḥdīd</u> (271:1) of

$$94;22,24^\circ .$$

102. Critique of the Main Determination

The plot on the fold-out gives the reader a bird's-eye view of the results of Bīrūnī's manifold computations, and enables him to make a visual estimate of their accuracy. The data are displayed on a rectangular grid in which the ratio between latitude and longitude scales has been so chosen that there is zero distortion along a parallel of latitude roughly bisecting the plot. There is, of course, no difference between the reckoning of modern and medieval latitudes. For the longitudes, it has been found convenient to convert those reckoned from Greenwich into those calculated from Bīrūnī's base meridian by adding twenty-six degrees. On the plot, modern coordinates are enclosed in parentheses to distinguish them from medieval ones. All the main localities (except Alexandria and Raqqa) mentioned in the text have been plotted, points fixed by modern coordinates being indicated by small circles, those from Bīrūnī's data by small triangles. All of the text coordinates have been rounded off to minutes. Legs of the various traverses are shown with heavy lines, the corresponding lines joining accurate determinations being thin and dotted.

The legs Jurjāniya-Balkh, Balkh-Ghazna, and Shīrāz-Zaranj are quite good, perhaps because in all three the terrain is relatively flat. The leg Rayy-Jurjāniya is conspicuously bad, the worst of the lot, the great circle distance having been grievously underestimated. The upshot of this is that the northern traverse (in spite of the distance Baghdad-Rayy being overestimated) drags Ghazna west of its true location. No single leg in the sourthern traverse is as badly off, so that the final result is better, especially the one in which the intermediate station at Bust has been eliminated.

For Bīrūnī's chosen Ghazna longitude of $94;22^\circ$, the $\Delta\lambda$ between it and Baghdad is $24;22^\circ$, the accurate amount being $24;2^\circ$.

Hence his error is a third of a degree in about twenty-four, or say, one part in seventy. Considering the extreme crudity of the terrestrial measurements at his disposal, the final result is very creditable indeed.

The plot shows the results of the two bad multiplication blunders (at 257:4 and 257:6) in grossly misrepresenting the position of Bukhārā. This, however, has no effect on the main result.

CHAPTER IX. CALCULATIONS OF THE QIBLA OF GHAZNA

103. The First Method (272:17 – 275:11)

Having determined the coordinates of the capital of the Ghaznavid empire. Bīrūnī crowns the achievement by calculating from them the azimuth of Mecca. He scorns the erroneous method of al-Battānī referred to in 233:6 and exhibits his virtuosity by a whole series of computations, all exact in theory. As will be seen below, however, small errors in each computation causes each result to differ slightly from all the others, and in fact, from the correct answer.

The procedure he follows with every case is to (1) give the rule, (2) carry out the computation and (3) prove the validity of the rule. We will break up the rule for each method into parts, applying and proving each part as we proceed. M and G stand for Mecca and Ghazna respectively.

First calculate

(273:2) $\quad \mathrm{Cos}\,\varphi_G \cdot \mathrm{Sin}\,\Delta\Lambda\,/\,R = \mathrm{Cos}\,\varphi_G \cdot \mathrm{Sin}(\Lambda_G - \Lambda_M)\,/\,R$

(271:1) $\quad = \mathrm{Sin}\,56;25^\circ \cdot \mathrm{Sin}(94;22,24^\circ - 67^\circ)\,/\,60$

(210:8) $\quad = \mathrm{Cos}\,33;35^\circ \cdot \mathrm{Sin}\,2[7];22,24^\circ\,/\,60$

(273:15) $\quad = 49;59,5 \times 27;35,14\,/\,60$

$\quad = 22;58,56 = \mathrm{Sin}\,p.$

(The first digit of the longitudinal difference at 273:14 should be restored to a 27 instead of the text's 26).

Hence

$$p = \mathrm{arc\,Sin}\,22;58,56 = 22;31,19^\circ\,,$$

where p is the "perpendicular" of 273:17.

To justify these computations, turn to Figure C65. On it O, the intersection of the horizon of Ghazna with the meridian of Mecca, is the pole of great circle EDZ. The situation is otherwise self-explanatory. Application of the Rule of Four to triangles EDT and HLT gives

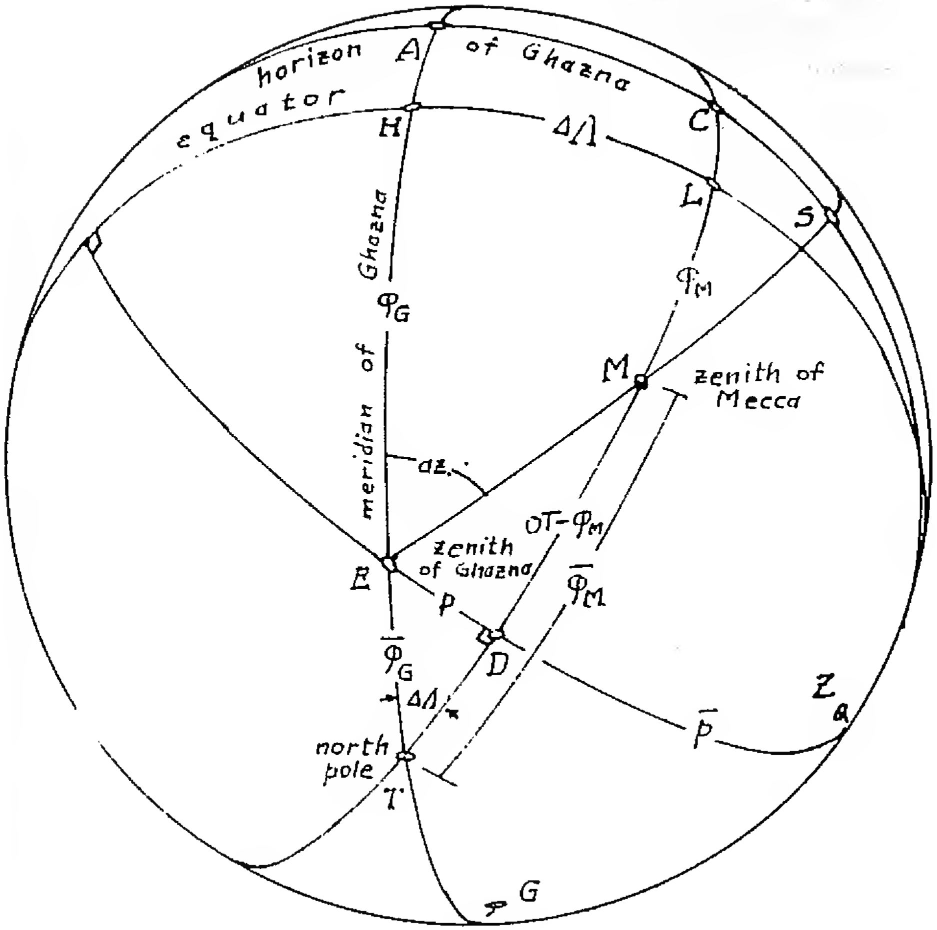


Figure C65

(275:18) $$\frac{\text{Sin } (TE = \bar{\varphi}_G)}{\text{Sin } (ED = p)} = \frac{\text{Sin } (TH = 90^o)}{\text{Sin } (HL = \Delta\Lambda)}$$

and this is equivalent to the expression above.

Now compute

(273:5) Sin $\varphi_G \cdot R$ / Cos p

= (Sin 33;35°) · R / Cos 22;31,19°

(273:18) = 33;11,20 × 60 / 55;25,26

= 1991;20,0 / 55;25,26

(273:20) = 35;55,44 = Sin OT = Sin $\overline{TD}$ = Cos TD.

Hence OT = arc Sin 35;55,44 = 36;46,48° .

There are two errors. The last digit of Sin OT should be 45, not 46, and the arc sine of the same number is in fact 36;47,16.

Proof that this is legitimate is somewhat involved. Bīrūnī was apparently ignorant of the relation

$$\cos a \cdot \cos b = \cos c$$

for a right spherical triangle with legs a and b and hypotenuse c. Application of this to triangle TED would have given our relation immediately. Instead he writes

(274:20) $$\frac{\text{Sin } (OT = \overline{DT})}{\text{Sin } (TG = \overline{ET} = \bar{\varphi}_G)} = \frac{\text{Sin } (OD = 90^o)}{\text{Sin } (DZ = \bar{p})}$$

This can be obtained by applying Menelaos' Theorem to the triangle TED cut by transversal OGZ, whence using the modern form of the theorem

$$\sin (EG = 90^o) \cdot \sin OT \cdot \sin ZD = \sin TG \cdot \sin OD \cdot \sin (ZE = 90^o).$$

Cancelling out the equal factors on the ends,

$$\sin OT \cdot \sin DZ = \sin TG \cdot \sin OD,$$

which is equivalent to expression 274:20, which is in turn equivalent to the expression for computation.

Now

(234:12) $\overline{OT - \varphi_M}$ = $\overline{36;46,48^o - 21;40^o}$

(273:21) = 90° − 15;6,48° = 74;53,12°

(275:4) = $\overline{DL - \varphi_M}$ = $\overline{MD}$ = MC.

Next calculate

(273:6) Cos MD · Cos p / R

= Sin MC · Sin $\bar{p}$ / R

= Sin 74;53,12° · Cos 22;31,19° / R

(274:3) = 3210;24 / 60 = 53;30,25

= Sin MS

Whence

MS = 63;5,54° .

In justification of this Bīrūnī writes

(275:5) Sin MC / Sin MS = Sin (CD = 90°) / Sin (ZD = $\bar{p}$),

which can be proved by applying the Menelaos Theorem (cf. Overview, p. 342) to triangle MED cut by transversal CSZ. Then

$$\sin MC \cdot \sin (SE = 90^o) \cdot \sin ZD = \sin CD \cdot \sin MS \cdot \sin (ZE = 90^o),$$

or, canceling equal factors,

$$\sin MC \cdot \sin ZD = \sin CD \cdot \sin MS,$$

which is equivalent to expression 275:5 and the computation.

Finally,

(273:8) Cos φ_M · Sin $\Delta\Lambda$ / Cos MS

= Cos 21;40° · Sin 2[7];22,24 / Cos 63;5,54°

(274:5) = 1538;17,11,24,6 / 27;8,51

= 56;39,50 = Sin az.

So

az. = 70;48,15° .

The validity of this follows from an application of the sine law to triangle TEM giving

(275:8) $$\frac{\text{Sin } (ME = \overline{MS})}{\text{Sin } (MT = \bar{\varphi}_M)} = \frac{\text{Sin } (\angle ETM = \Delta\Lambda)}{\text{Sin } (TEM = 180^o - \text{az.})},$$

since an arc and its supplement have equal sines.

We note that ~~that~~ all the relations employed involve angles and arcs on the surface of the sphere – never entities within its interior.

104. The Second Method (276:1 - 279:8)

This particular example was first investigated by Miss Varsenig T. Khachadourian. In the presentation which follows we again mingle rule, numerical example, and proof, whereas the text keeps them separate. Constant reference will be made to Figure C66. Note that in it Bīrūnī again makes use of the "time triangle" and "day triangle" already introduced in Section 59.

The first step is to calculate

(276:2) $(\text{Cos}\,\Delta\varphi\,)\cdot R\,/\,\text{Cos}\,\varphi_G = \left[\text{Sin}\,(\overline{\varphi_G - \varphi_M})\right]\cdot R\,/\,\text{Cos}\,\varphi_G$

(273:12) $= \left[\text{Cos}\,(33;35^\circ - 21;40^\circ)\right]\,60\,/\,\text{Cos}\,33;35^\circ$
(234:12)
(277:2) $= 58;42,25 \times 60\,/\,49;59,5$

$= 70;28,12 = HT,$

called in the translation the "diameter" (from qutr). "Hypotenuse" would be a more suitable translation (also called qutr), for the result measures the hypotenuse of the day triangle. That this is a fact follows from the proportion

(278:18) $$\frac{HD\ (=\text{Cos}\,GH = \text{Cos}\,\Delta\varphi\,)}{HT} = \frac{\text{Sin}\,(\angle HTD = \bar{\varphi}_G\,)}{(\text{Sin}\,\angle\,HDT) = R}\,.$$

Next, compute

(276:5) $\text{Sin}\,\Delta\Lambda\cdot\text{Cos}\,\varphi_M/R$

(274:15) $= \text{Sin}\,27;22,24^\circ\cdot\text{Cos}\,21;40^\circ\,/\,60$

$= 27;35,14 \times 55;45,39\,/\,60$

(277:7) $= 1538;1711,24,6\,/\,60$

$= 25;38,17 = FE = OM,$

the "modified sine of the longitude (difference)". This follows from the fact that

$$OM = \text{Sin}_{LM}\,\Delta\Lambda = \frac{LM}{R}\,\text{Sin}\,\Delta\Lambda = \frac{\text{Cos}\,\varphi_M\cdot\text{Sin}\,\Delta\Lambda}{R}$$

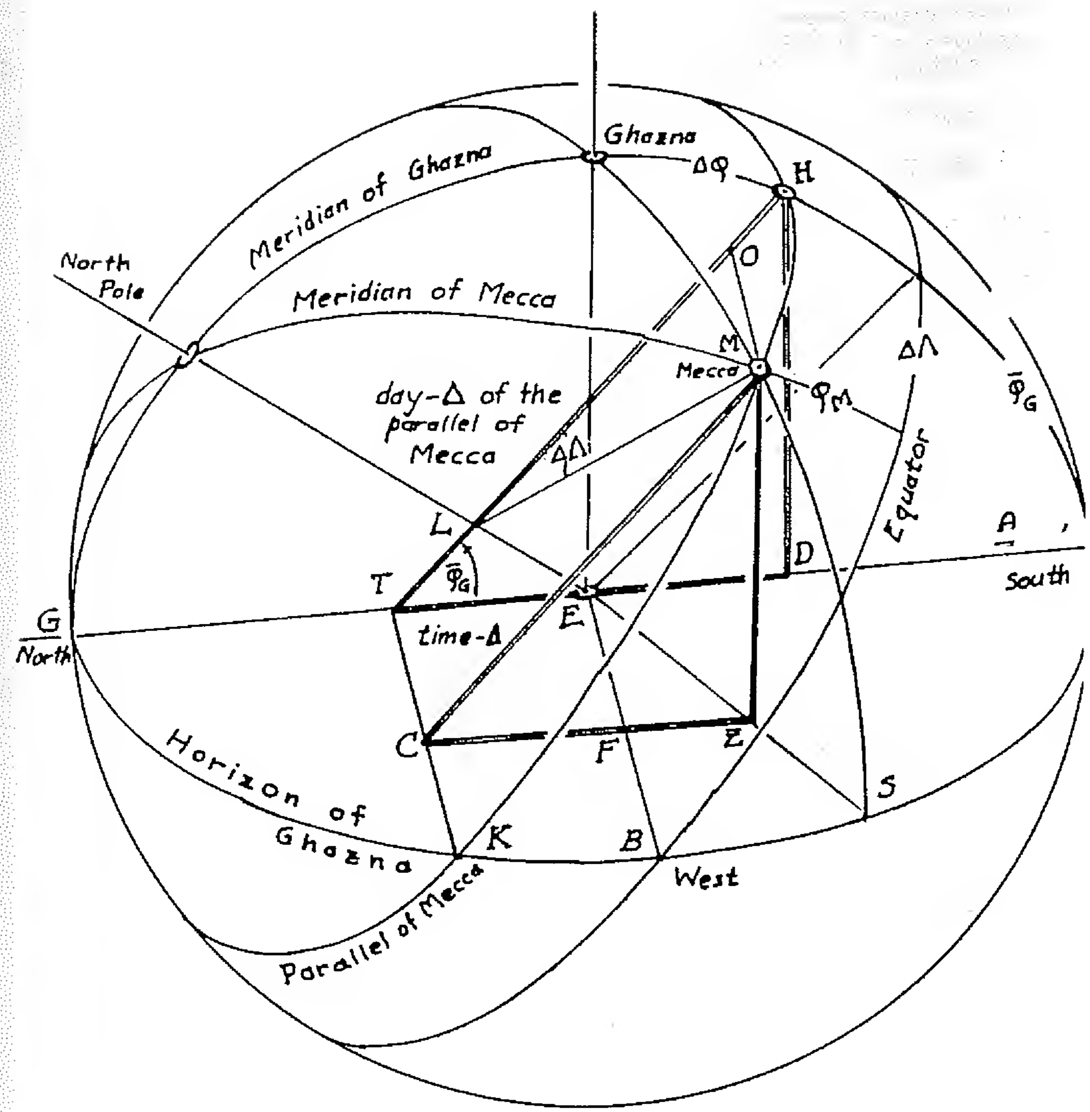


Figure C66

Now obtain

(276:5) $\text{Vers } \Delta\Lambda \cdot \text{Cos } \varphi_M / R$

(274:15) $= \text{Vers } 27;22,24^\circ \cdot \text{Cos } 21;40^\circ / 60$

(277:5) $= 6;43,9 \times 55;45,39 / 60$

$= 374;39,58,47,51 / 60$

(277:8) $= 6;14,40 = HO = \text{Vers}_{LM} \Delta\Lambda$,

the "modified versed sine of the longitude (difference)", which is demonstrable by the same type of argument exhibited just above. In this calculation the last digit of Vers $\Delta\Lambda$ should be 6, not 9.

Next calculate the "remainder" (the hypotenuse of the time-triangle

(277:9) $MC = TO = HT - HO$

$= 70;28,12 - 6;14,40$

$= 64;13,32$.

Then form

(276:8) $MC \cdot \text{Sin}\varphi_G / R = 64;13,32 \times \text{Sin } 33;35^\circ / R$

$= 2131;34,29,22,40 / 60$

(277:11) $= 35;31,34 = CZ$,

the "retained amount". This follows from the proportionality

(278:20) $MC / CZ = (\text{Sin}\angle MZC = R) / \text{Sin}(\angle CMZ = \varphi_G)$.

Also calculate

(276:11) $(\text{Sin } \varphi_M) \cdot R / \text{Cos } \varphi_G$

$= (\text{Sin } 21;40^\circ) \cdot R / \text{Cos } 33;35^\circ$

$= 22;9,8,32 \times 60 / 49;59,5$

(277:13) $= 26;35,27 = ET = CF$,

the "gauge". In the text the fourth digit of Sin φ_M shown by us above has been truncated, a very small error. The validity of the computation follows from the proportion

(279:2) $(EL = \text{Sin } \varphi_M) / ET = \text{Cos } \varphi_G / R.$

The truth of the criteria

(276:12) if the gauge $ET <$ the retained CZ the azimuth is south,

and if $ET > CZ$ the azimuth is north,

can be inferred from the figure.

Next

$|CZ - ET| = 35;31,34 - 26;35,27$

(277:15) $= 8;56,7 = FZ$,

and

(276:14) $\sqrt{FZ^2 + FE^2} = \sqrt{8;56,7^2 + 25;38,17^2}$

(277:16) $= \sqrt{79;50,21,4,49 + 657;18,35,36,49}$

$= 27;9,1, = EZ$,

where the last digit of the root has been truncated.

Finally

(276:12) $EF \cdot R / EZ$

$= 25;38,17 \times 60 / 27;9,1$

(277:19) $= 56;39,29 = \text{Cos } \widehat{BS} = \text{Sin } \widehat{AS} = \text{Sin az.}$

and $\text{az.} = \text{arc Sin } 56;39,29 = 70;47,13^\circ$.

Here, in contrast to Method 1, the operations are performed upon rectilinear configurations inside the sphere. This, combined with the use of the uncommon versed sine function and the peculiar day and time triangles, creates the impression that the method was old-fashioned in Bīrūnī's day, perhaps inherited from the earliest period of Islamic astronomy which was strongly influenced by the Indians.

105. The Third Method (279:9 – 282:13)

This approach, originally studied by Mr. Gayzag Boyajian, utilizes some of the computations of the preceding method. Take the "modified versed sine" (OH in Figure C66, HM = YO in C67) and use it to form

(279:12) $\quad HM \cdot \mathrm{Sin}\,\varphi_G / R$

(277:8, 273:18) $\quad = 6;14,40 \times 33;11,20 / 60$

$\quad = 207;14,46,13,20 / 60$

(280:10) $\quad = 3;27,15 = MC = DL,$

the above references being to Figure C67. This follows from the proportion

(282:18) $$\frac{HM\ (= \mathrm{Vers}_{HA}\ \Delta\Lambda)}{MC} = \frac{\mathrm{Sin}\,(\angle HCM = 90^\circ)}{\mathrm{Sin}\,(\angle MHC = \varphi_G)}.$$

(Note that both text and translation are wrong here and should have "HMC" restored to "MHC" and "colatitude of Ghazna" to "latitude of Ghazna").

Now

$AD = \mathrm{Vers}\,(\bar{\varphi}_G + \varphi_M)$

(281:16) $\quad = \mathrm{Vers}\,(56;25^\circ + 21;40^\circ)$

(273:12, 234:12) $\quad = \mathrm{Vers}\ 78;5^\circ$

(280:11) $\quad = 47;36,39,$

where the last digit should be 38, not 39.

Also $\quad AL = AD + DL$

$\quad = 47;36,39 + 3;27,15 = 51;3,54,$

which is called the "gauge". It is different from the magnitude given the same name in the preceding section, but serves the same purpose. If it is sufficiently small that L falls between E and A the azimuth is south; if L is between E and G it is north.

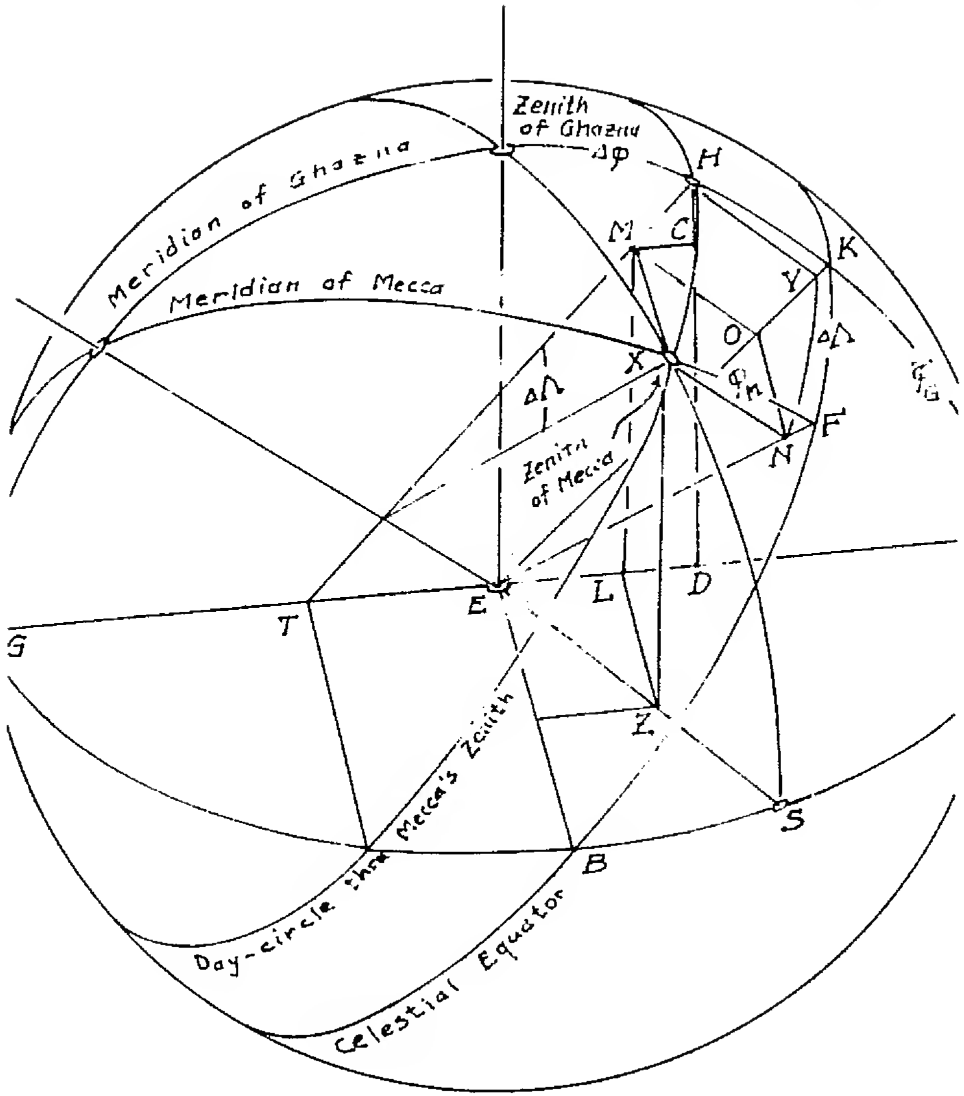


Figure C67

105. The Third Method (279:9 – 282:13)

This approach, originally studied by Mr. Gayzag Boyajian, utilizes some of the computations of the preceding method. Take the "modified versed sine" (OH in Figure C66, HM = YO in C67) and use it to form

(279:12) $HM \cdot \text{Sin}\,\varphi_G / R$

(277:8, 273:18) $= 6;14,40 \times 33;11,20 / 60$

$= 207;14,46,13,20 / 60$

(280:10) $= 3;27,15 = MC = DL,$

the above references being to Figure C67. This follows from the proportion

(282:18) $$\frac{HM\ (=\text{Vers}_{HR}\ \Delta\Lambda)}{MC} = \frac{\text{Sin}\,(\angle HCM = 90^\circ)}{\text{Sin}\,(\angle MHC = \varphi_G)}.$$

(Note that both text and translation are wrong here and should have "HMC" restored to "MHC" and "colatitude of Ghazna" to "latitude of Ghazna").

Now

$AD = \text{Vers}\,(\bar{\varphi}_G + \varphi_M)$

(281:16) $= \text{Vers}\,(56;25^\circ + 21;40^\circ)$

(273:12, 234:12) $= \text{Vers}\ 78;5^\circ$

(280:11) $= 47;36,39,$

where the last digit should be 38, not 39.

Also $AL = AD + DL$

$= 47;36,39 + 3;27,15 = 51;3,54,$

which is called the "gauge". It is different from the magnitude given the same name in the preceding section, but serves the same purpose. If it is sufficiently small that L falls between E and A the azimuth is south; if L is between E and G it is north.

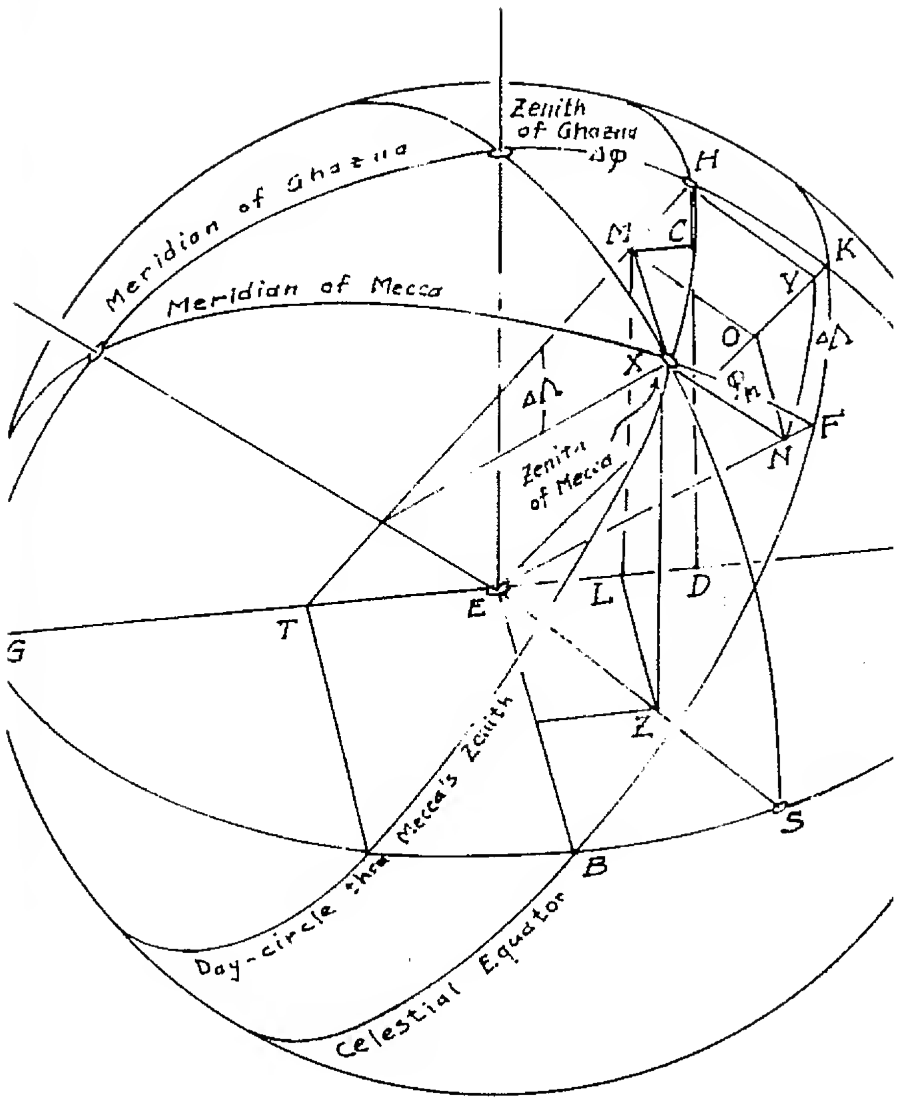


Figure C67

The "modified sine" $MX = LZ = \mathrm{Sin}_{RX}\Delta\Lambda$ (OM in Figure C66) was calculated in 277:7 above. Also

$$EL = EA - AL$$

(280:13) $$= 60 - 51;3,54 = 8;56,6.$$

Hence

$$EZ = \sqrt{LZ^2 + EL^2}$$

(277:7) $$= \sqrt{25;38,17^2 + 8;56,6^2}$$

$$= \sqrt{657;18,35,36,49 + 79;50,3,12,36}$$

(280:15) $$= \sqrt{737;8,38,49,25}$$

$$= 27;8,41.$$

The square root is somewhat off; the accurate value is 27;9,1,26. Finally, as in the preceding section,

(282:12) $$LZ \cdot R / EZ = 25;38,17 \times 60 / 27;8,41$$

$$= 56;40,11 = \mathrm{Sin}\ AS$$

(280:17) $$= \mathrm{Sin\ az.},$$

and $$\mathrm{az.} = 70;49,16^{\circ}$$

A trivial variant of this method is to calculate

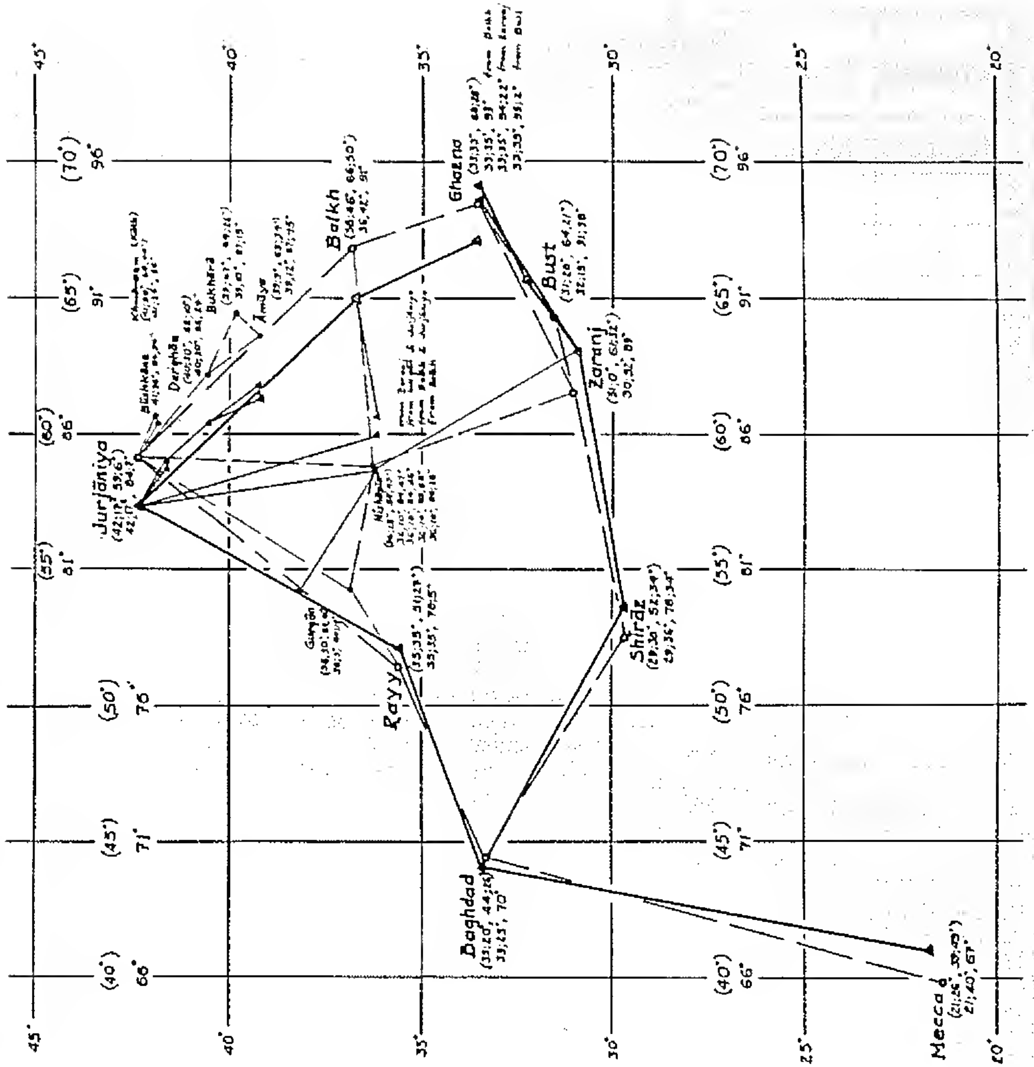
(282:14) $$LZ \cdot R / EL = 25;38,17 \times 60 / 8;56,6$$

$$= 1538;17,0 / 8;56,6$$

(282:19) $$= 172;9,50 = \mathrm{Tan\ az.}$$

Whence

$$\mathrm{az.} = 70;47,9^{\circ}.$$



No. of Obs.	OBSERVER	PLACE	① LONGITUDE (from the Canaries; add 10° to the Atlantic norm)	② WEEK-DAY and LOCAL TIME P.M.	DATE, MONTH, and YEAR of NAB.	③ ΔΛ = 104;22,24° − ① in degrees thence × 0;10 in day-minutes	④ LOCAL TIME in day-min. = ② + 2;30	⑤ WEEK-DAY and GHAZNA TIME of the equinox = ③ + ④ (if neg. add 60 and delete a day)	⑥ JULIAN DAY-NO. and DATE (at the place of the observation)
1 297:2	Hipparchus	Rhodes	60;30° (297:4)	Tues. 6ʰ	30, XI 586	43;52,24° 7;18,44	15	Tuesday 22;18,44	1662 522 27 Sept. 162 B.C.
2 297:4	"	"	"	Sat. −6ʰ	1, epag. 589	" "	−15	Friday 52;18,44	1663 618 27 Sept. 159 B.C.
3 297:12	"	"	"	Sun. 0ʰ	1, epag. 590	" "	0	Sunday 7;18,44	1663 983 27 Sept. 158 B.C.
4 297:16	"	"	"	Sun. −12ʰ	4, epag. 601	" "	−30	Saturday 37;18,44	1668 001 27 Sept. 147 B.C.
5 297:19	"	"	"	Mon. 6ʰ	4, epag. 602	" "	−15	Sunday 52;18,44	1668 366 27 Sept. 146 B.C.
6 [illegible]	"	"	"	Thurs. 6ʰ	4, epag. 605	" "	15	Thursday 22;18,44	1669 461 26 Sept. 143 B.C.
7 298:4	Ptolemy	Alexandria	"	Wed. 2ʰ	7, III 880	" "	5	Wednesday 12;18,44	1769 539 25 Sept. 132 A.D.
8 298:10	"	"	"	Fri. −5ʰ	9, III 887	" "	−12;30	Thursday 54;48,22	1772 096 26 Sept. 139 A.D.
9 298:14	Yahyā Cf. Section 20	Baghdad	80° (294:3)	Sun. 0;48ʰ	25, VIII 1577	24;22,24° 4;3,44	2;0	Sunday 6;3,44	2024 112 19 Sept. 829
10 299:3	anon.	"	"	Mon. −1ʰ	25, VIII 1578	" "	−2;30	Monday 1;33,44	2024 477 19 Sept. 830
11 299:4	(From a treatise by Thābit.) Cf. Section 11	"	"	Tues. 7ʰ	25, VIII 1579	" "	17;30	Tuesday 21;33,44	2024 842 19 Sept. 831
12 299:9	Khālid Cf. Section 20	Damascus	70° (299:13)	Thurs. −12;48	26, VIII 1580	34;22,24° 5;43,44	−32;0	Wednesday 33;43,44	2025 207 18 Sept. 832
13 299:16	anon.	Baghdad	80° (294:3)	Thurs. 9;22ʰ(−24)	29, VIII 1591	24;22,24° 4;3,44	23;25	Wednesday 27;28,44	2029 226 20 Sept. 843
14 300:3	al-Makkī Cf. Section 23	Nīshāpūr	95° (266:4)	[Sat.] 0ʰ	30, VIII 1599	9;22,24° 1;33,44	0	[Saturday] 1;33,44	2032 147 19 Sept. 851
15 300:9	Banū Mūsā Cf. Section 15	Samarra	79;45° (300:11)	Tues. 0ʰ	2, IX 1607	24;37,24° 4;6,14	0	Tuesday [4];6,14	2035 069 19 Sept. 859
16 300:13	al-Battānī Cf. Section 22	Raqqa	73° (294:2)	Wed. 13;15ʰ	8, IX 1630	31;22,24° 5;13,44	33;7,30	Tuesday 38;21,14	2043 470 19 Sept. 882
17 301:1	Ibn ʿIṣma Cf. Section 23	Balkh	101° (261:1)	Wed. 1;36ʰ	9, IX 1636	3;22,24° 0;33,44	3;0	Wednesday 3;[3]3,[4]4	2045 661 18 Sept. 888
18 301:6	al-Sufī Cf. Section 24	Shīrāz	88;33,32° (264:11)	Sun. 1ʰ	29, IX 1718	15;48,52° 2;38,8,40	−2;30	Sunday [0];3,8,40	2075 611 18 Sept. 970
19 301:9	"	"	"	Mon. 6ʰ	29, IX 1719	" "	15	Monday 17;38,8,40	2075 976 18 Sept. 971
20 301:12	Abū al-Wafāʾ Cf. Section 24	Baghdad	80° (294:3)	Fri. −3ʰ	30, IX 1722	24;22,24° 4;3,44	−7;30	Thursday 56;34,44	2077 072 18 Sept. 974
21 302:1	al-Bīrūnī	Jurjāniya	84;0,54° (250:16)	Mon. 1ʰ	10, X 1764	10;21,30° 1;43,35	2;30	Monday 4;13,35	2092 412 17 Sept. 1016
22 302:6	"	Ghazna	104;22,24° (271:1)	Thurs. 19;0ʰ	10, X 1767	0 0	47;30	Thursday 47;30	2093 507 17 Sept. 1019

The last digit of the result should be 11, not 9. A second modification puts

(283:2) $\qquad EL \cdot R / LZ = 8;56,6 \times 60 / 25;38,17$

$\qquad = 536;6,0 / 25;38,17$

(283:7) $\qquad = 20;54,37 = \text{Cot az.},$

and $\qquad \text{az.} = 70;47,11^{\circ},$

which is what should have been obtained just above. In fact, accurate interpolation for the arc cotangent gives 10 in the last digit, not 11.

Medieval cotangent tables tended to be computed with R = 12 (the "digits" of 283:13) rather than 60 (e.g. the Khwārizmī Zīj, p. 174, and Battānī, vol. 2, p. 60). For the benefit of readers checking his results from such tables Bīrūnī switches parameters in the cotangent above, obtaining

$$\text{Cot}_{12}\,\text{az.} = \frac{12}{60}\,\text{Cot}_{60}\,\text{az.} = 0;12 \times 20;54,27$$

(283:15) $\qquad = 4;10,55$

These two variants are of interest as being the only example in the entire book where Abū Rayḥān avails himself of the shadow functions (tangent and cotangent).

The general technique resembles that of Method 2.

106. An Analemma Construction for the Qibla (289:1–9)

The magnitude shown as a three-dimensional representation in Figure C67 are easily laid off in a single plane to yield a graphical construction for the qibla in terms of φ_G, φ_M, and $\Delta\Lambda$. Assuming that the cardinal directions, say EA and EB in Figure C67.1, have been determined in a suitable horizontal circle, it will suffice to find the magnitudes EL and LZ (Figure C67) to determine the arc AS, the azimuth of the qibla.

≡ King's instrument

Both analemma / Proj. circle / straight line

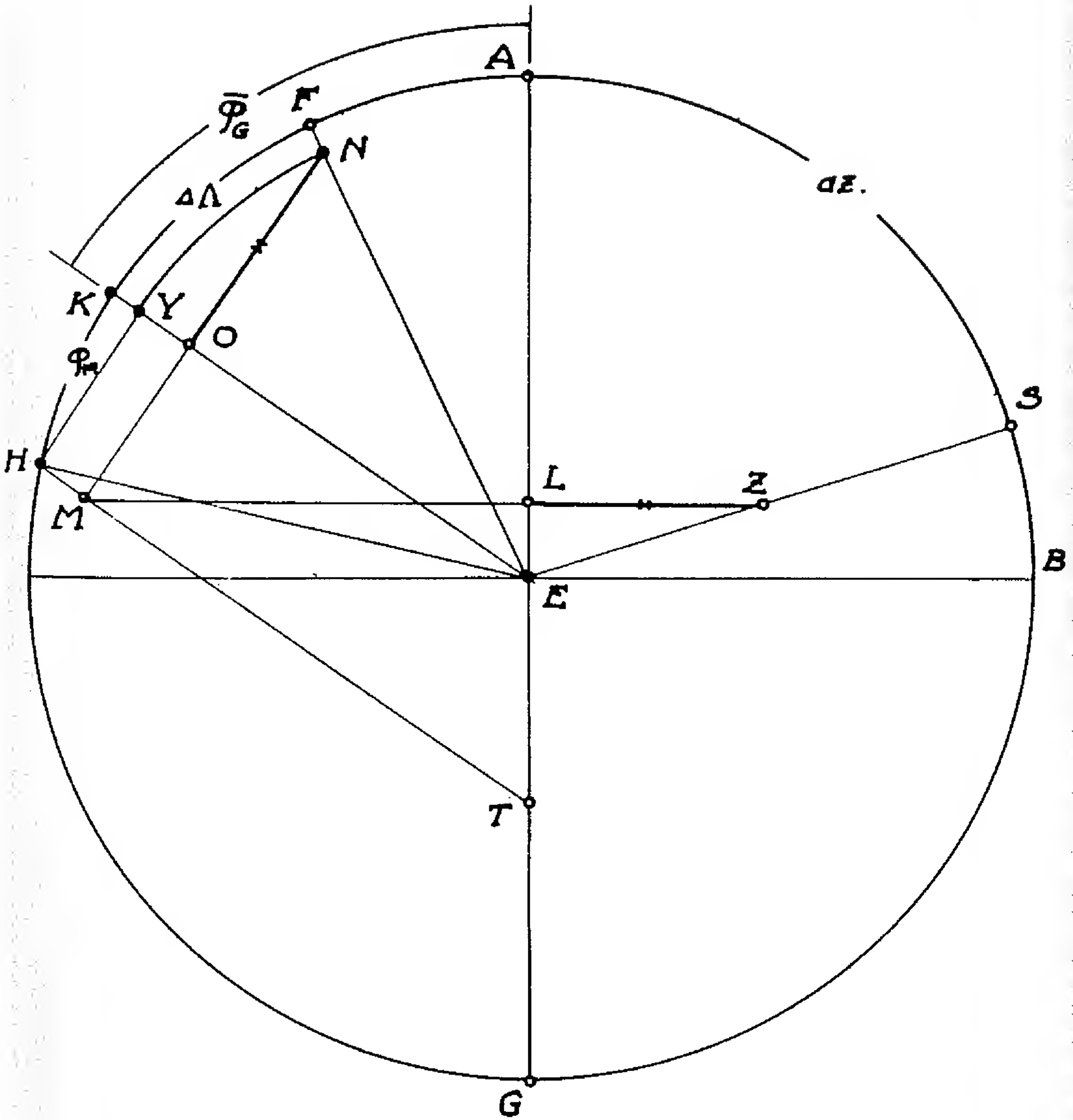


Figure C67.1

From the south point, A, lay off arc AK equal to $\bar{\varphi}_G$, then KH = φ_M, and on the other side, KF = $\Delta\Lambda$. If, then, HT is drawn parallel to KE, it will represent a portion of the diameter of the day-circle of Mecca. Further, if a perpendicular is dropped from H to KE, the segment from its foot, Y, to the center, E, will be equal to the radius of this day-circle. With center E and radius EY describe an arc meeting FE at N. This arc will be of magnitude $\Delta\Lambda$, and will equal the arc of the day-circle comprehended between the meridians of Ghazna and Mecca, the arc XH. Hence the perpendicular NO dropped from N to KE will equal in magnitude the distance from X, the zenith of Mecca, to the meridian plane of Ghazna. And this magnitude, XM, in turn equals LZ, one of the two segments needed to solve the problem.

It remains to find EL. But the latter is equal to the distance (on Figure C67) from M to the vertical line through E. We proceed to locate M on Figure C67.1, the analemma, by extending ON, the perpendicular to KE, until it meets HT at the desired M: Thence, dropping a perpendicular from M to AE, the foot of the perpendicular is L. Lay off from L on this perpendicular, and in proper direction, LZ = ON. From E draw the radius ES through Z. The arc AZ is the required azimuth.

There exist in the literature several analemmas for the qibla, one by Ibn al-Haitham (fl. 1000) having been studied and published (see Schoy, Alhazen and Schoy, Kibla). Bīrūnī himself was involved with at least two other such constructions, one in the Qānūn (p. 526), and another described in the unpublished Leiden Cod. Or. 168(16), ff. 136r-137r. The latter is a discussion of a solution by Ḥabash (cf. Section 37) which resembles the construction given here. not the same

107. The Fourth Computational Method (284:10 - 286:1)

This final example, like the first, operates entirely on the surface of the sphere. It was first investigated by Mr. Haratiun Arsen. Working from Figure C68, it is easy to verify that the Rule of Four gives

(284:12) $\quad \text{Sin}\,(TM = \bar{\varphi}_M) \,/\, \text{Sin}\,MK \;=\; \text{Sin}\,(TL = 90^{\circ}) \,/\, \text{Sin}\,(LH = \Delta$

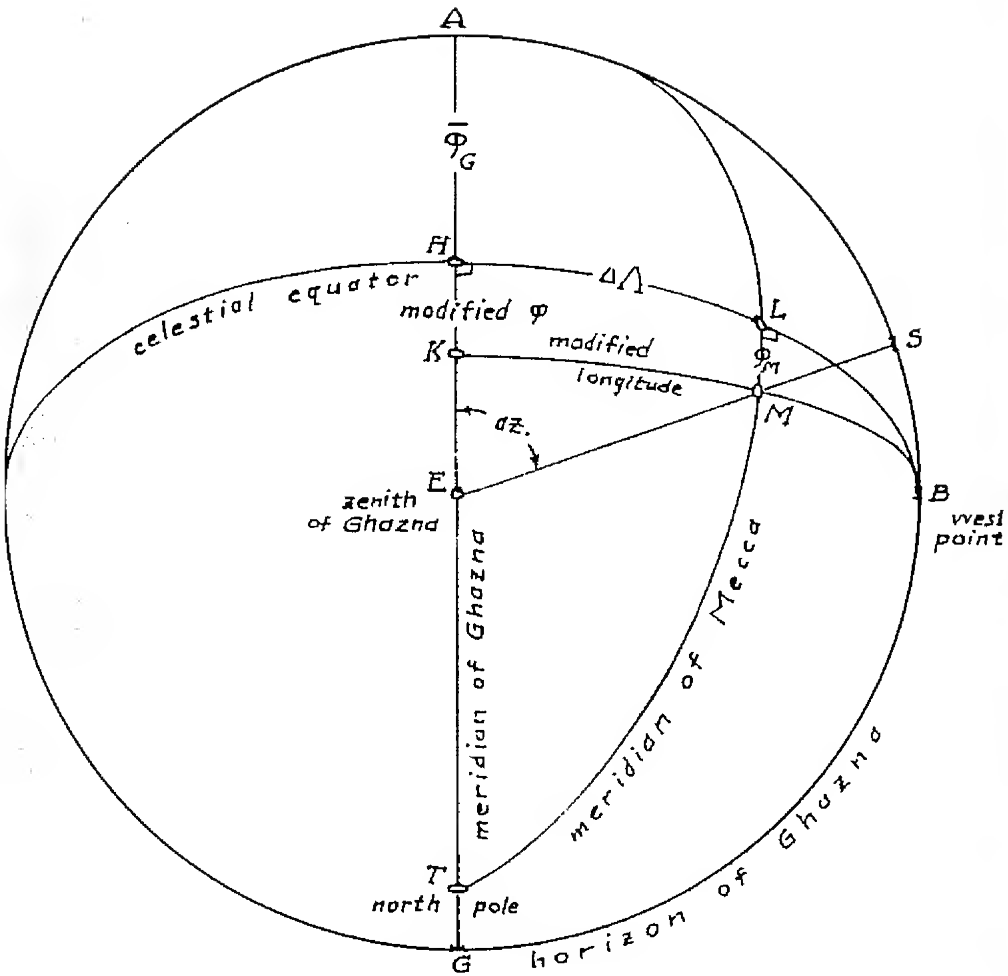


Figure C68

hence Sin MK can be calculated. For our example it is equal to the line OM on Figure C66, previously calculated as 25;38,17 at 277:7 whence arc MK is found to be 25:17,47° at 285:4. Now apply the sine law to triangle BML to obtain

(284:14) $$\frac{\text{Sin }(BM = \overline{MK})}{\text{Sin }(ML = \varphi_M)} = \frac{\text{Sin }(BK = 90^\circ)}{\text{Sin }KH}$$

From this,

$$(\text{Sin }\varphi\ \ )\cdot R / \text{Cos } MK$$

$$= (\text{Sin } 21;40^\circ)\cdot R / \text{Cos } 25;17,47^\circ$$

(277:12) $$= 1329;8,0 / 54;14,48$$

$$= 24;30,6 = \text{Sin } KH,$$

whence

(285:7) $$KH = 24;6,7^\circ,$$

which is called the "modified latitude".

Now, $KE = HE - HK = \varphi_G - HK$

(285:8) $$= 33;35^\circ - 24;6,7^\circ = 9;28,53^\circ.$$

Bīrūnī next sets down the relation

(284:17) $$\frac{\text{Sin }(BM = \overline{KM})}{\text{Sin }(MS = h_M\ )} = \frac{\text{Sin }(BK = 90^\circ)}{\text{Sin }(KA = \overline{KE})},$$

which can be justified by applying the Menelaos Theorem to the triangle EKM cut by the tranversal ASB. This gives the equality

$$\text{Sin } BM \cdot \text{Sin } KA \cdot \text{Sin }(ES = 90^\circ) = \text{Sin } BK \cdot \text{Sin }(EA = 90^\circ)\cdot \text{Sin } MS,$$

which is equivalent to expression 284:17.

So

(285:9) $\text{Cos KE} \cdot \text{Cos KM} / R$

$= \text{Cos } 9;28,53^\circ \cdot \text{Cos } 25;17,47^\circ / 60$

(285:4) $= 59;10,49 \times 54;14,48 / 60$

$= 3210;19,58,5,12 / 60$

$= 53;30,19 = \text{Cos EM},$

in which the last digit should be 20, not 19. Thence

$$EM = \bar{h} = 26;54,20^\circ .$$

This is converted into miles

(285:11) $26;54,20 \times 56\frac{2}{3} = 1524;38,53,$

thence into farsakhs.

Finally, by the Rule of Four,

(284:19) $\text{Sin EM} / \text{Sin MK} = \text{Sin } (ES = 90^\circ) \text{ Sin } (SA = az.),$

so $(\text{Sin MK}) \cdot R / \text{Sin EM}$

(277:7) $= (\text{Sin } 25;17,4^\circ)60 / \text{Sin } 26;54,20^\circ$

$= 25;38,17 \times 60 / 27;9,4$

(285:12) $= 56;39,23 = \text{Sin az.},$

whence az. $= 70;46,56^\circ$.

108. Rational Approximation to the Qibla of Ghazna – Evaluation of the Results (286:2–12)

As the text states, the directions in this passage are sufficiently simple that they may be carried out by unlettered workmen.

The first rule is equivalent to putting

(286:7) Cos az. $= 1/3 = 0;20,0,0,$

whence az. $= 70;31,44^\circ$.

The more precise one has

(286:10) Sin az. $= 1 - \frac{1}{18} = \frac{17}{18} = 0;56,40,0,$

whence az. $= 70;48,43^\circ$.

In order to check the accuracy of these rules we have recomputed the qibla precise to three sexagesimal places, obtaining

$$az. = 70;47,6^\circ .$$

So the first approximation is about a quarter of a degree off, and the second only about a minute and a half, both good enough for practical purposes.

For ease of comparison we assemble below all six of the results obtained in the text:

			error
The First Method	(274:5)	$70;48,15^\circ$	$0;1,9^\circ$
The Second Method	(277:19)	$70;47,13^\circ$	$0;0,7^\circ$
The Third Method	(280:17)	$70;49,16^\circ$	$0;2,10^\circ$
tangent variant	(282:19)	$70;47,9^\circ$	$0;0,3^\circ$
cotangent variant	(283:7)	$70;47,11^\circ$	$0;0,5^\circ$
The Fourth Method	(285:12)	$70;46,56^\circ$	$-0;0,10^\circ$

In four out of the six cases the error is ten seconds or less. The errors are in all cases random, and have no significance in so far as passing judgment upon the methods is concerned. The second and third methods involve seven nontrivial computations each; the first and fourth four each. Hence from the standpoint of the labor involved the latter two are preferable.

The subject of qibla computations was, understandably, a favorite of the Islamic astronomers. Practically all zījes give rules for the determination, and many of them have several. A survey of the history of and relations between the many techniques involved would be a project of considerable interest, but it has not thus far been undertaken. Our only contribution along these lines is the conjecture made at the end of Section 104. Pending a complete investigation, the reader is referred to Schoy, Bestimmung; Schoy, Kibla; and Schoy, Nairīzī in the bibliography.

109. Graphical Determination of the Meridian (286:13 - 290:13)

Before it is possible to lay out the direction of the qibla at an actual site, it is essential to have the cardinal directions. A classical method of determining the north-south line in medieval times was to describe a circle on a horizontal plane, and at its center to erect a vertical gnomon of suitable length. Where the endpoint of the gnomon's shadow crossed the circle during the forenoon the place was marked. In like manner the point also was marked where the lengthening shadow extended itself beyond the circle in the afternoon. The meridian line was then taken as the diameter bisecting the arc between the two marks. This technique was known as the Indian Circle (287:1).

In lieu thereof Bīrūnī describes an elegant analemma construction which calls for only one solar observation, although at the time the observer must note not only the shadow direction, but also h, the solar altitude. He must also have at hand the solar declination at the time, δ, and φ, the local latitude. But an individual in possession of this knowledge and the azimuth of the qibla could quickly ascertain the direction of prayer anywhere in the vicinity he might find himself, provided the sun were visible. The method differs only trivially from one he explains in the Canon (p. 449, cf. Merid.)

The construction itself is self-explanatory, and has been reproduced on Figure C70. Its validity is easily demonstrated with the aid of Figure C70.1, which show the situation in space. From it we note that the angle the meridian line makes with the shadow direction is the acute angle of a right triangle of which the hypotenuse is Cos h, and the leg adjacent to the angle in question equals the difference between the base of the time triangle (cf. Section 59) and Sin r, where r is the rising amplitude. Hence it will suffice to show that in Figure C70 KLE is equal to this triangle, and its angle at E is the angle referred to. KEL is a right triangle, for angle L is inscribed in a semicircle.

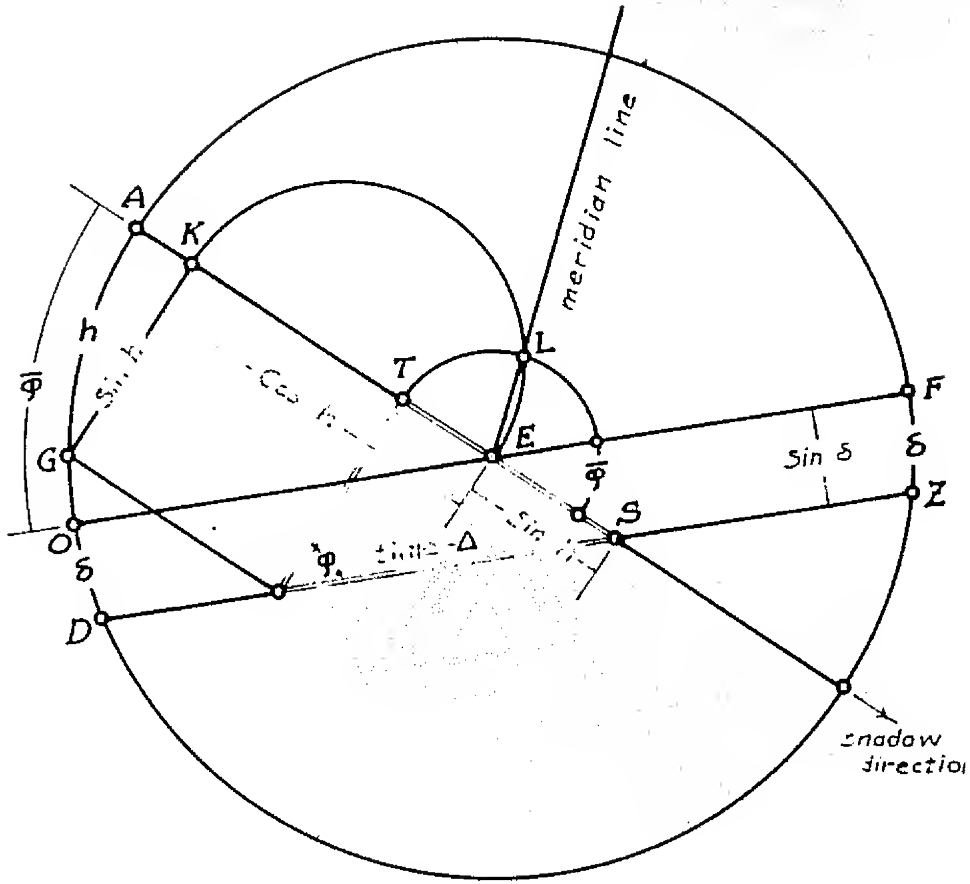


Figure C70

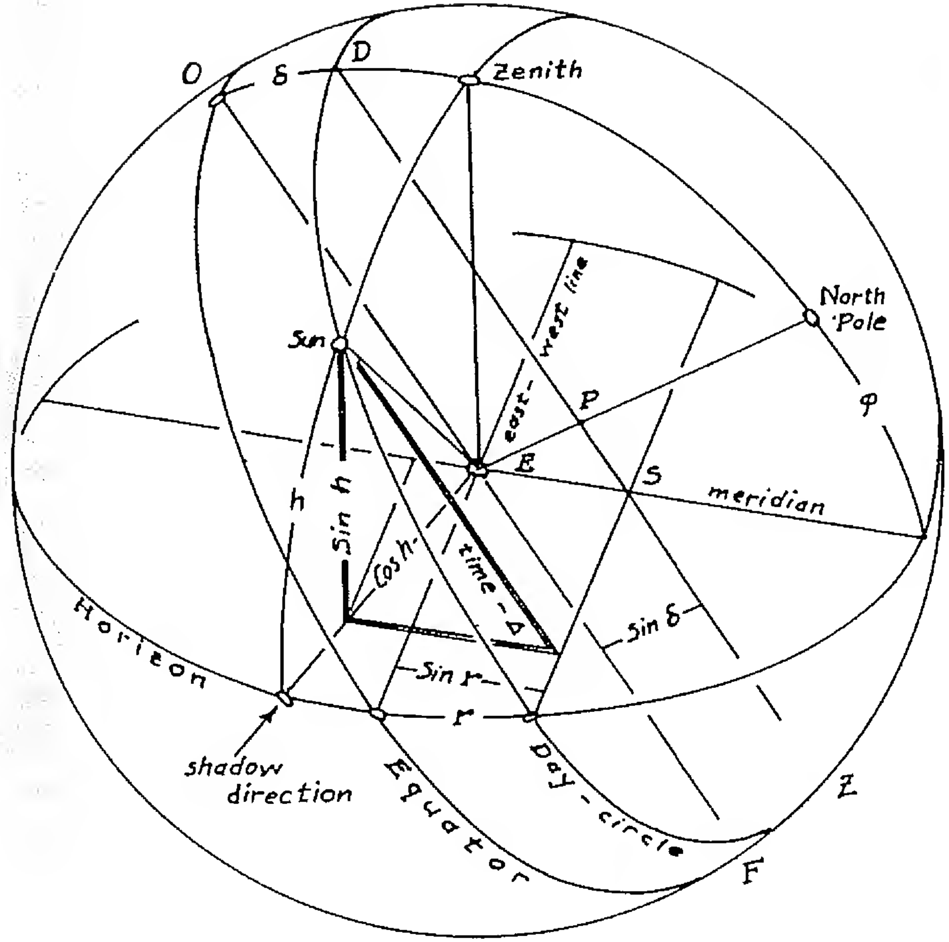


Figure C70.1

Furthermore, its hypotenuse has been constructed as Cos h. As for the leg LE, it is equal to TE. The triangle HTS is indeed equal to the time triangle, for its altitude is Sin h, and the angle adjacent to its altitude is φ. It remains only to show that ES = Sin r as marked. Returning to Figure C70.1, we observe that in triangle EPS

$$\cos \varphi = EP / ES = \mathrm{Sin}\, \delta / \mathrm{Sin}\, r,$$

so

$$\mathrm{Sin}\, r = \mathrm{Sin}\, \delta / \cos \varphi .$$

But in Figure C70,

$$\sin \bar{\varphi} = \mathrm{Sin}\, \delta / ES,$$

so

$$ES = \mathrm{Sin}\, \delta / \sin \bar{\varphi} = \mathrm{Sin}\, \delta / \cos \varphi ,$$

thus showing that ES = Sin r and completing the proof.

Having now exhausted the subject of the qibla, Bīrūnī rounds off the discussion, just as he had previously prefaced it, with a statement of the benefits conferred by scientific knowledge. He remarks that the determination of azimuths and meridians is as important for the Jews, Christians, and Sabians (Section 4) as for the Muslims.

Bīrūnī's attitude toward astrology expressed in the following passage (289:16 – 290:14) is equivocal. Be this as it may, he is right in asserting that a scientific knowledge of geography is essential to a serious application of the subject. Astrological predictions are based upon a calculation of the horoscope – the celestial longitude of the ecliptic point rising across the eastern horizon at a given instant. The result will be affected not only by the apparent time at the locality for which it is cast, but also by the local latitude. In general, two horoscopes cast for the same time but different localities will be different.

The term Hashtmarj (290:13) is made of two words. Hasht is Persian for "eight", and marj means "meadow". It evidently denotes some variety of necromancy, but is unknown to us.

CHAPTER X. MORE LONGITUDE COMPUTATIONS; EQUINOX OBSERVATIONS

110. Two Eclipse Observations (290:14 – 292:5)

In this passage the author belabors those astronomers of Khurāsān who are adherents of the Sindhind tradition (cf. Section 29), citing their errors in the prediction of two eclipses which Bīrūnī himself observed. One difficulty arose from confusion concerning the two common base meridians discussed in Section 48 above. Depending on which base is used, the longitude of Baghdad was taken as either 70° or 80°. In Battānī's zīj the latter is found, and this is consonant with the value of $\Lambda = 73^\circ$ ($73;15^\circ$ in our version, vol. II, p. 41) for Raqqa, the town for which the tables of the zīj were computed. Somehow or other the Khurāsānīs chose 70° for Baghdad, thus introducing the error of 10° ($= 10^\circ / 15^{\circ/h}$) $= \frac{2}{3}^h$ of which Bīrūnī writes in 291:8.

The first eclipse was lunar, Oppolzer No. 3439, with maximum immersion at $23;13^h$ universal (Greenwich) time on 16 September, 1019. The sun was then at Virgo $28;50^\circ$, or thereabouts (Tuckerman, p. 527). Converting to Ghazna time, we estimate that first contact took place at about $2;15^h$ on the morning of 17 September, i.e., about eight hours after sunset the previous evening, the middle of the eclipse occurring at about $3;45^h$, final clearance would be at about eleven and a half hours after sunset. All this generally confirms Abū Rayḥān's remarks in the passage 291:11 – 20.

The second eclipse, a solar one, was Oppolzer No. 5285. The time given for it converts to about 6;50 A.M. Ghazna time, well after sunrise for that season. Again the validity of Bīrūnī's remarks are confirmed.

He says that Lamghān, the place from which be observed this eclipse, is between Qandahār and Kābul, i.e. southwest of the latter. Nevertheless, it seems reasonably clear that the town is the modern Laghman, about thirty miles east and slightly north of Kābul. Lamghān is mentioned in the India (transl., vol. I, pp. 259, 317), and its coordinates and those

of Kābul are given in the Canon (p. 574). They are:

	Λ	φ Canon	φ India
Lamghān	$96;10^\circ$	$33;50^\circ$	$34;43^\circ$
Kābul	$95;30^\circ$	$33;45^\circ$	$33;47^\circ$

These are about the relative positions of Laghman and Kābul.

Galen (fl. 170 A.D.) was "the last great medical writer in Greek antiquity" (Eine, vol. II, pp. 402–3). The work referred to in 292:6 was probably the essay entitled "That the Best Physician is also a Philosopher" (Kieffer, p. 2).

111. The Longitude Difference Between Baghdad and Raqqa (292:6 – 294:23)

In locating Ghazna longitudinally with respect to Mecca, Bīrūnī has thus far used Baghdad as a sort of base locality. He now feels that he should fix Ghazna with respect to the bases adopted in commonly used zījes. These are:

(1) The Cupola (293:4), base locality of the Sindhind, concerning which see Section 67 above. As between it and Baghdad,

(209:8) $\Delta\Lambda = 90^\circ - 70^\circ$

(293:6) $= 20^\circ / 15 \text{ hours} = 1\frac{1}{3}^h$.

Dropping the seconds of arc in the longitude of Ghazna adopted at 271:1, between Ghazna and the Cupola is

$$\Delta\Lambda = 94;22^\circ - 90^\circ$$

(293:7) $= 4;22^\circ = 4 + \frac{1}{5} + \frac{1}{6}$ degrees

$$= 4;22^\circ / 15 = 4;22 \times 0;4$$

$$= 9;17,28^h$$

$$\approx 0;17 = \frac{15+2}{6} = \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{10} \text{ hours.}$$

(2) The Almagest and Handy Tables, the latter known to the medieval astronomers as Theon's Canon, are both based upon Alexandria. Bīrūnī attributes to the Almagest a $\varphi = 30;58^{\circ}$ for Alexandria, and between it and Babylon a

$$\Delta\Lambda = (\frac{1}{2} + \frac{1}{3})^{h} = 0;50^{h} = 0;50 \times 15 \text{ degrees} = 12;30^{\circ} \ (293;11).$$

Both citations are correct, being from Almagest V, 12 and IV, 6 respectively. However, in both the Handy Tables and the Geogr. this $\Delta\Lambda$ is $18;30^{\circ}$ and the latitude of Alexandria is $31;0^{\circ}$. He states that "modern" astronomers put it at

$$\Delta\Lambda = 13\frac{3}{4}^{\circ} = 13\frac{3}{4}^{\circ} / 15 \text{ hours} = (55 / 4 \times 15)^{h} = 0;55^{h}$$

$$= \frac{11}{12} \text{ hours} = (\frac{1}{2} + \frac{1}{4} + \frac{1}{6})^{h}.$$

Baghdad is close to Babylon, but because of the discrepancy in the above he postpones relating Ghazna to Alexandria until making the independent computations discussed below.

(3) Raqqa is the base of Battānī's zīj. Located on the Euphrates, almost due west of Mawṣil (Mosul), in Abbasid times it was one of the principal towns of Upper Mespotamia (LeStr., 101; cf. Section 22). From the table of geographical coordinates in Battānī he notes (294:1) the following longitudes:

Alexandria	$60;30^{\circ}$
Raqqa	73°
Babylon	79°
Baghdad	80°

The published version of the zīj confirms all of these, save that it has $73;15^{\circ}$ for Raqqa. We note that Battanī's base meridian runs through the Canaries (cf. Section 48), hence his longitudes tend to be 10° more than Abū Rayḥān's.

Subtracting the longitude at the head of the column from the three following, we obtain for the $\Delta\Lambda$ between Alexandria and the places named:

Raqqa	$12;30^{\circ}$	(also 10° elsewhere in Battānī, vol. III (text), p. 63.)
Babylon	$18;30^{\circ}$	
Baghdad	$19;30^{\circ}$.	

To have an independent basis for assessing these figures, Bīrūnī calculates $\Delta\Lambda$ between Baghdad and Raqqa by an application of the trapezoid algorism of Section 77. The two latitudes are:

(203:18) $\varphi_1 = 36;10^{\circ}$ for Raqqa, and

(203:19) $\varphi_2 = 33;25^{\circ}$ for Baghdad,

(294:9) whence $\Delta\varphi = 2;36^{\circ}$.

$$\widehat{AB} = 110 \text{ farsakhs} = 110 \times 3 \text{ miles} = 330 / 56\frac{2}{3} \text{ degrees}$$

(294:14) $= 5;49,34^{\circ}$,

where the last digit is wrong. It should be 25.

$$\text{Crd } AB = 6;5,54,$$

which also has an error in the last digit; 56 is correct. Hence

$$\Delta\Lambda = \text{arc Crd}\left[\sqrt{(\text{Crd}^2\, 5;49,34^{\circ} - \text{Crd}^2\, 2;36^{\circ}) \text{ Cos } 33;25^{\circ} / \overline{\text{Cos } 36;1^{\circ}} \cdot R / \text{Cos } 33;25^{\circ}}\right]$$

(294:9, 238:8)

$$= \text{arc Crd}\left[\sqrt{(6;5,54^2 - 2;43,21^2) \times 50;4,52 / 48;31,51 \times 60 / 50;4,52}\right]$$

$$= \text{arc Crd } (\sqrt{30;43,43,59,26} \times 60 / 50;4,52)$$

(294:19) $= \text{arc Crd } 6;38,28 = 6;20,43^{\circ}$

where the last digit of the final result should be 42.

So

$$\Lambda_{\text{Raqqa}} = \Lambda_{\text{Baghdad}} - \Delta\Lambda$$

(294:3) $= 80^{\circ} - 6;20,43^{\circ}$

$= 73;39,17^{\circ}$.

In the report of al-Hāshimī, discussed in Section 66, the $\Delta\Lambda$ is 7°.

112. The Longitude Difference Between Raqqa and Alexandria (295:1 – 296:4)

In like manner, the author indulges in a last application of the trapezoid algorism to check the $\Delta\Lambda_{\text{Raqqa-Alexandria}}$.

Now

(203:18) $\varphi_1 = 36;1^\circ$ for Raqqa, and

(204:11) $\varphi_2 = 30;58^\circ$ for Alexandria,

whence

(295:4) $\Delta\varphi = 5;3^\circ$,

And $\widehat{AB} = 628$ miles

$$= 628 \;/\; 56\tfrac{2}{3} \text{ degrees}$$

(295:9) $= 11;4,56^\circ$.

The value of Crd AB, given as 11;31,4, is badly off. It should be 11;35,14.

Now

$$\Delta\Lambda = \text{arc Crd}\left[\sqrt{(\text{Crd}^2\, 11;4,56^\circ - \text{Crd}^2\, 5;3)\ \text{Cos}\ 30;58^\circ \;/\; \text{Cos}\ 36;1^\circ \cdot R \;/\; \text{Cos}\ 30;58^\circ}\right]$$

(295:11)
$$= \text{arc Crd}\left[\sqrt{(11;31,4^2 - 5;17,12^2) \times 51;26,53 \;/\; 48;31,51 \times 60 \;/\; 51;26,53}\right]$$

$$= \text{arc Crd}\left[\sqrt{104;42,37,17,5[2] \times 51;26,53 \;/\; 48;31,51 \times 60 \;/\; 51;26,53}\right]$$

$$= \text{arc Crd}\left[\sqrt{111;0,16,27,49} \times 60 \;/\; 51;26,53\right]$$

(295:15) $= \text{arc Crd}\ 12;17,14 = 11;45,15^\circ$.

The restoration in the text's last digit in the difference between the squares is from 56 to 52, a scribal error. That this is so is clear from two considerations. First, the restoration corrects the error in the subtraction operation. Second, only if the restoration is made will the product of the difference in the squares and Cos φ_2 come out as shown in the text.

The result is

$$\Lambda_{\text{Raqqa}} = \Lambda_{\text{Alexandria}} + \Delta\Lambda$$

(294:2) $= 60;30^\circ + 11;45,15^\circ$

$= 72;15,15^\circ$,

or

$$\Lambda_{\text{Alexandria}} = \Lambda_{\text{Raqqa}} - \Delta\Lambda$$

(294:2) $= 73^\circ - 11;45,15^\circ$

(296:4) $= 61,14,45^\circ$.

113. Local Time Difference (296:5–18)

The author is now in a position to settle the matter of the time difference between Ghazna and four of the localities mentioned in Section 111 above. For Ghazna he tacitly uses a longitude of $104;22,24^\circ$, adding 10° (for the change of base meridian) to the value settled upon in 271:2. We tabulate his results below, recalling that an hour's time difference corresponds to 15° of daily rotation. Since in the following section he uses also the unit of "day-minutes", sixtieths of a day, in our last column the time differences have been converted into these units. Since $360^\circ = 24^h = 60$ day-minutes, to convert from degrees into day-minutes multiply by $60 / 360 = 0;10$. To change from hours into day-minutes multiply by $60 / 24 = 2\frac{1}{2}$. For day-minutes Bīrūnī uses (297:7) the term <u>jharī</u>, a transliteration of the Sanskrit <u>ghaṭī</u>. Elsewhere he transliterates the first letter with a <u>kāf</u> (or <u>gāf</u>, e.g. <u>India</u>, transl., vol. I, p. 337; <u>Canon</u>, p. 77; <u>Shadows</u>, 126:16, etc.)

	①	②	③	④
	(from the Canaries)	$\Delta\Lambda$ = 104;22,24° − ①	Hours difference = ②/15	Day-minutes difference = ② × 0;10
Alexandria	60;30°	43;52,24°	2;55,29,36	7;18,44
Raqqa	73°	31;22,24°	2;5,29,36	5;13,44
Baghdad	80°	24;22,24°	1;37,29,36	4;3,44
The Cupola	100°	4;22,24°	0;17,29,36	0;43,44

The results in the text corresponding to our column ② are rounded to minutes (or perhaps Bīrūnī sensibly dropped the seconds from the Λ beforehand).

114. Autumnal Equinox Observations (297:1 – 302:15)

Bīrūnī now applies the material just calculated by listing all autumnal equinox observations known to him and converting them to Ghazna time. The information he gives has been systematized and laid out in our chart (see verso of the fold-out). The last column gives the Julian day number and Julian date of the observation. It is clear from the context that noon epoch is being used. Hence if the equinox occurred before noon at Ghazna, Column 5 shows the week-day preceeding that of Column 2. Nevertheless the terminology is sometimes confusing. For instance, at 299:7 "eight hours after the beginning of Wednesday" means "eight hours after sunrise on Wednesday". In all cases, being at equinoxes, sunrise and sunset are at 6 A.M. and 6 P.M. respectively.

Later on, in the Canon (p. 640), Bīrūnī reported again essentially the same information, but with the addition of a twenty-third observation. This was the autumnal equinox of 1020, observed by him at Ghazna.

The zīj of Ibn Yūnus (the part published in Caussin) reports Observations 4, 8 through 14, and 16. Observations 4, 8, 10, and 11 are reported also by Thābit (p. 269).

The source for the first eight observations is Almagest III, 1.

Concerning Observation 9, Ibn Yūnus ascribes to Yaḥyā an equinox at the same time of day as Bīrūnī reports, but he does not give the week-day, and the date is 25 Murdād, 199 Yazdigird (= 19 September 830), a year after our date. Restoration of a single digit in Ibn Yūnus' report, 198 for 199, would make the two records coincide. This seems much preferable to changing Bīrūnī's, since the latter would involve altering not only the year, but also the week-day. Moreover both documents would then purport to give two different results from the same place for the autumnal equinox of 830.

Ibn Yūnus states that our Observation 10 was reported by Thābit, which is indeed the case. Both of them give the equivalent Persian date (25 Murdād, 199 Yazdigird), and both state that the time was "seven hours of the day", i.e. 1 P.M., in contrast to Bīrūnī's "one hour before midday".

For Observation 11, both Thābit and Ibn Yūnus give the equivalent date and the same time of day as Bīrūnī.

For Observation 12 Ibn Yūnus names Sanad b. ʿAlī as well as Khālid. He gives the equivalent date, but for the time of day he has 28;15 day-minutes (= 11;18^h) P.M. This differs slightly from Bīrūnī's "twelve hours and four fifths of an hour before noontime" (= 24^h − 12;48^h = 11;12^h P.M.)

For Observation 13 (299:16) the text gives for the Ghazna time Wednesday, 28 Pharmuthi, whereas the Baghdad time is said to be Thursday the twenty-ninth. We see no way to reconcile this apparent inconsistency other than to read at 299:17 "three hours and one fifth and one sixth from the even of Thursday". Ibn Yūnus gives the same hour, but for the equinox of a year later.

For Observation 14 (300:4) the translation has Sunday, but the text has Saturday, correctly. Essentially the same information is given by Ibn Yūnus, who adds that the measurement was made in the presence of Ṭāhir b. ʿAbdallāh. This Ṭāhir II was a grandson of the founder of the Ṭāhirid dynasty (see Section 23 above).

At 300:10 in Observation 15, the translation should read "noontime of Tuesday, the second of the month of Pachons". The Ghazna time of day works out at 4;6,14 rather than 13;6,14 as in the text and translation. The first digit in this number indeed looks like a 13 in the MS, but it can also be read as a 4, and is best so read.

For Observation 16 (300:13), to reconcile the Tuesday, 7 Pachons, Ghazna time and the Wednesday, 8 Pachons, Raqqa time, read at 300:14 "from the beginning of the eve of Wednesday, the eighth of Pachons". Ibn Yūnus correctly quotes the zīj of al-Battānī as giving the same time of day as reported by Bīrūnī, the date being 19 Aylūl of the year 1194 of (Alexander) the Two-Horned. Provided one takes (with Nallino, Battānī, vol. I, pp. 42, 209-210) these years as beginning in September (Aylūl), the equivalent Julian date is 19 September 882, the same as that implied by the Taḥdīd.

At 301:2 read "seven hours and three fifths of an hour from sunrise on Wednesday". For this observation, 17, our resulting time of day at Ghazna is 3;33,44 day-minutes. The characters of the text taken as 3;43,14 can be restored to this without much violence.

The Ghazna time of day for the next observation we obtain as 0;8,8,40, whereas the text has 5;8,8,40. This is a case of confusing the Arabic sexagesimal zero symbol with the letter ha' (= 5), which it resembles.

At 301:13 for "three hours from the beginning of Friday" read "three hours after sunrise on Friday".

At 302:2 "seven hours from the beginning of Monday" means seven hours after sunrise, i.e. 1 P.M.

The colophon at 302:13 notes that the copying of our unique manuscript of the Taḥdīd was completed at Ghazna c. 20 September, 1025. Thus there is every reason to think that Bīrūnī, just having completed his fifty-second year, oversaw the copying of the book. However, we agree with Bulgakov (Taḥdīd, text, pp. 15-16) that it is not a holograph.

115. Computational Technique in the Taḥdīd

The typical medieval mixture of sexagesimal and decimal representations, the former predominating, is found in this book. Except where they are simple, and easily reducible to sums of unit fractions, fractions, and the fractional parts of mixed numbers are shown as sexagesimals in the Arabic alphabetical numeral forms. In transcribing them we separate sexagesimal digits by commas, and employ the customary semicolon for a sexagesimal point, for which there is no analogue in the text. The integer parts of numbers are also represented by the Arabic letter-numerals, but in non-place-value decimals. In general, however, if the integer part becomes large, with three or more digits, Bīrūnī switches over to Indian numerals in the Eastern Arabic form still current in the Near and Middle East.

For actual computations two traditions are utilized. In the first half of the book (say through 155:23) multiplication, division, and root-taking are carried out by first expressing the numbers involved as decimal integers of the proper denomination, performing the required operations upon these integers, and then converting the final result back into sexagesimal form. Thus the sexagesimal 2;5,29 is written as $(2 \times 60^2) + (5 \times 60) + 29 = 7529$ seconds. Of course it is necessary to modify the denominations properly as the computation proceeds: seconds times thirds give fifths, the square root of fourths is in seconds and so on. This technique was doubtless predominant among the scientists of the Middle Ages.

For the main train of longitude computations, however, throughout the second half of the book, all indications are that these operations were performed upon sexagesimals. In general, all partial results are displayed, and they are sexagesimal except for the integer parts.

By and large, Bīrūnī seeks and attains precision to seconds of arc. The trigonometric tables in his Canon, written after the Taḥdīd, have entries to four and five significant sexagesimal digits, and he may have them already at hand for the Taḥdīd. In general he rounds off correctly, and it is impossible to say where a particular rounding error has been caused by faulty tables. Like Ptolemy, however, Bīrūnī was insensitive to the effects of combining numbers calculated to different orders of precision.

Thus, at 223:2 he introduces into the same computation one number with a single significant sexagesimal digit, and a second number with four! Later in the same computation he operates with a number having the equivalent of nine decimal digits. But at 81:1 he evinces general distrust of results arrived at by long trigonometric computations, and states his preference for direct observations involving minimum numerical reduction.

Computational errors detected in the text have been pointed out in the commentary at the place of occurrence. Bīrūnī seems to have shared with other great scientists a penchant for confining his few bad mistakes in calculation to sections of the work which were of minor significance (cf. Section 88 above).

In general, his approach to numerical problems would not seem foreign to a reasonable reader living in any age or place. A curious exception to this general rule is his implicit use of linear zigzag functions (Sections 21 and 27) where we would automatically invoke functions of smooth variation.

116. Trigonometry in the Taḥdīd

The book contains much material of latent relevance to the history of trigonometry. But the reader examining it with this end in view must bear in mind that Bīrūnī's objective in writing the Taḥdīd was to do mathematical geography, not, as in the Maqālīd, to describe the historical development of trigonometric methods. Therefore, although he usually proves the validity of his solutions, he feels no obligation to ease the task of the future historian by naming the theorems he uses, much less to ascribe them to authors or dates.

The leading impression conveyed by an examination of the work is the author's almost exclusive reliance on the sine (and cosine) function. This was certainly not from ignorance of the tangent function (or the cotangent, secant, and cosecant), for he devoted an entire book (the Shadows) to this subject. There is only one passage in the Taḥdīd where he actually computes with the tangent (Section 105), and then it is introduced incidentally, almost as an afterthought, or to satisfy people with tangent tables on their hands. The situation is the more astonishing since there are places (e.g. 76:5) where use of the tangent would have shortened the computations considerably.

Occasional application is made of the versed sine function (Sections 59 and 104), presumably in contexts taken over from Indian astronomy and associated with the curious "time-triangle" and "day-triangle". Here Bīrūnī's nomenclature is unorthodox; he calls the function (219:13, 228:11) jayb al-maʿkūs, the "reversed sine" rather than the usual sahm (= Latin sagitta) "arrow".

The trapezoid algorism used for calculating longitude differences (Section 77) employs the ancient chord function, the ancestor of the sine, but this can hardly be called trigonometry.

It is of interest to note that by the time Bīrūnī got around to writing the Canon he had adopted a completely modern definition for the sine by putting $R = 1$. In the Taḥdīd he maintains the traditional $R = 60$.

Although nowhere mentioned as such, the author's workhorse trigonometric relation is the "Rule of Four Quantities" (cf. Overview) the theorem which states that in a pair of spherical right triangles having a pair of acute angles equal (say $A = A'$),

$$\operatorname{Sin} a / \operatorname{Sin} c = \operatorname{Sin} a' / \operatorname{Sin} c' ,$$

where capital letters designate angles, the cognate small letters denote the sides opposite them, and $C = C' = 90^\circ$.

On one occasion (at 163:8) the tangent case of the Rule of Four is employed, which asserts that in the triangles referred to above,

$$\operatorname{Tan} a / \operatorname{Sin} b = \operatorname{Tan} a' / \operatorname{Sin} b' .$$

The only other relation frequently called upon is the sine law, which uses the fact that in any spherical triangle

$$\operatorname{Sin} a / \operatorname{Sin} A = \operatorname{Sin} b / \operatorname{Sin} B = \operatorname{Sin} c / \operatorname{Sin} C.$$

Usually in the Taḥdīd the law is applied to right triangles, but in one passage (271:15) oblique triangles are involved.

In three situations (Sections 103 and 107) Bīrūnī invokes an equation equivalent to the identity

$$\cos c = \cos a \cdot \cos b$$

which holds for all right spherical triangles. It seems clear from the context, however, that he did not have this relation as such, but obtained its equivalent by an application of the Menelaos Theorem, which dates from Hellenistic times.

These topics about exhaust the trigonometric resources of the Tahdīd. As is invariably the case with medieval astronomical writings, the lack of negative numbers causes a tedious multiplicity of special cases and figures.

117. Calendrical Remarks

The epoch of the Muslim (or Hijra) calendar according to the popular reckoning is taken as Friday, 16 July, 622 A.D. (Julian Day 1,948,440). This was the evening of first visibility of the lunar crescent marking the first day of the pagan Arab year in which the Prophet Muḥammad emigrated to Medina from Mecca.

According to the astronomical reckoning the epoch is the preceding day, Thursday, 15 July, the day of the true new moon, the conjunction, at which time the crescent was invisible. (Cf. Ginzel, p. 252).

When a day of the week is cited along with a particular Muslim date, it is possible to determine which epoch has been used. Such dates are found in the passages of our text listed below:

	Text	Commentary Section	Observer	Epoch
1	75:9	17	al-Bīrūnī	popular
2	80:2	17	al-Bīrūnī	astronomical
3	86:8	18	Damascene records	popular
4	91:15	21	Khālid	popular
5	93:4	21	Khālid	popular
6	94:13	22	Banū Mūsā	astronomical
7	94:14	22	Banū Mūsā	astronomical
8	95:8	22	Banū Mūsā	astronomical
9	95:11	22	Banū Mūsā	astronomical
10	96:7	23	Ibn ʿIṣma	astronomical
11	96:9	23	Ibn ʿIṣma	astronomical
12	98:12	23	al-Hirawī	popular
13	99:11	24	al-Ṣūfī	popular
14	99:18	24	al-Ṣūfī	popular
15	101:4	25	al-Kūhī	astronomical
16	102:13	27	al-Khujandī	astronomical
17	102:19	27	al-Khujandī	astronomical
18	119:1	32	al-Bīrūnī	astronomical
19	119:14	32	Ibn al- Amīd	astronomical
20	120:15	32	al-Bīrūnī	popular
21	129:16	36	al-Bīrūnī	popular
22	130:6	37	al-Bīrūnī	popular
23	149:4	47	Ibn al-Ṣabbāḥ	astronomical
24	203:11	66	al-Hāshimī	popular

We note that neither of the two usages predominates. All the observers listed seem consistent in using one epoch or the other, except Bīrūnī himself, who employs both.

In the reckoning of dates in the Persian (or Yazdigird) calendar there is also a minor divergence of usage. The 365-day year was composed of twelve thirty-day months plus five epagomenal days. According to one custom these five days were inserted at the end of the year. A second doctrine put them immediately following Ābān, the eighth month (Ginzel, p. 287). In some of the passages listed above, the Persian date of the event as well as the Muslim date is given, and of these, for entries 1, 10, 13, and 17 the date falls after Ābān of the particular year. For all of these the epagomenal days follow Ābān; they are not left to the end of the year.

BIBLIOGRAPHY

Aaboe & Price

Asger Aaboe and Derek J. Price, Qualitative Measurement in Antiquity, Melanges Alexandre Koyre Paris, 1964, vol. 1, pp. 1-2

Almagest

Ptolemäus, Handbuch der Astronomie, Deutsche Übersetzung und erläuternde Anmerkungen von K. Manitius, Vorwort and Berichtigungen von O. Neugebauer, 2 vols., Leipzig, 1963.

Aristotle, see De Caelo
Meteorol.

Āryabhaṭīya

Āryabhaṭa, Āryabhaṭīya, ed. H. Kern, Leiden 1874; transl. W. E. Clark, Univ. of Chicago Press, 1930.

Battānī, zīj of

Al-Battānī sive Albatenii Opus Astronomicum, ed. and transl. by C. A. Nallino, 3 vols., Milan, 1899-1907.

Bīrūnī, see Canon
Chron.
India
Maqālīd
RT
Shadows
Tafhim
Tahdīd

Boilot

D. J. Boilot, L'oeuvre d'al-Beruni, essai bibliographique, Mélanges de l'Institut Dominicain d'Etudes Orientales, 2(1955), pp. 161-256.

Bourdon

Claude Bourdon, Anciens canaux, anciens sites et ports de Suez, Société Royale de géographie d'Egypte, Cairo, 1925.

Canon

Al-Bīrūnī, Al-Qānūn'l-Mas'ūdī (Canon Masudicus), 3 vols., Hyderabad-Dn., 1954-19.

Captains

Rudyard Kipling, Captains Courageous, London, 1927.

Caussin

Caussin de Perceval, Le Livre de la grande Table Hakémite . . . par Ebn Iounis . . . , Notices et extraits des mss. de la bibl. nationale . . . , tome septième, Paris, an XII de la république.

Chords

Al-Bīrūnī, Maqāla fī istikhrāj al-awtār fī l-dāira . . . , extant in four MSS. The Patna version was published as the first of four treatises in a single volume entitled Rasā'ilu l-Bīrūnī, Hyderabad-Dn., 1948. The Leiden version was translated into German by H. Suter in Bibliotheca Mathematica, vol. 11(1910), pp. 11-78. Both versions were used in a Russian translation by S. A. Krasnova and L. A. Karpova with a commentary by B. A. Rosenfeld and S. A. Krasnova in Iz istorii nauki i tehniki v stranah Vostoka, vyp. III, pp. 93-1963, Moscow.

Chron.

Al-Bīrūnī, Al-āthār al-bāqiya min al-qurūn al-khāliya, ed. by E. Sachau as "Chronologie orientalischer Voelker von

Alberūnī", Leipzig, 1878; reprinted Leipzig, 1923; reprinted Baghdad, 1963. Transl. by E. Sachau as, "The Chronology of Ancient Nations", London, 1879.

Comm. Vol.

Al-Bīrūnī Commemoration Volume, Calcutta, 1951.

Daffiná

Paolo Daffiná, Aral, Caspio, Tanais, Rivista degli Studi Orientali, XLIII (1968), pp. 363-378.

Dante

The Comedy of Dante Alighieri the Florentine, Cantica II, Purgatory, transl. by Dorothy L. Sayers, Penguin Books, Baltimore, 1955.

Davidian

Marie-Louise Davidian, Al-Bīrūnī on the Time of Day from Shadow Lengths, Journal of the American Oriental Society, vol. 80 (1960), pp. 330-335.

De Caelo

Aristotle, On the Heavens, text w. transl. by W. K. C. Guthrie (Loeb Classical Library), London, 1939.

DSB

Dictionary of Scientific Biography, New York, 1970 —.

EI

The Encyclopaedia of Islam, 4 vols., Leyden and London, 1913-1934.

EIne

The Encyclopaedia of Islam, New edition, Leiden and London, 1960 —.

Exact Sc.

O. Neugebauer, The Exact Sciences in Antiquity, 2nd ed., Providence 1957.

Fazārī

David Pingree, The Fragments of the Works of al-Fazārī, Journal of Near Eastern Studies, vol. 29 (1970), pp. 103-123.

Abū al-Fidā

Geographie d'Aboulfeda traduite de l'arabe en francais . . . , Paris, vol. I and vol. II, pt. 1 by Reinaud, 1848; vol. II, pt. 2 by Guyard, 1883.

GAL

Carl Brockelmann, Geschichte der arabischen Literatur, 2nd., Leiden, 1943. G1, G2, S1, and S2 indicate volumes and supplementary volumes respectively. Page references are to the second edition.

Geogr.

Claudius Ptolemaeus, Geographia, ed. C. F. A. Nobbe, Leipzig, 1843-45. An edition with Latin translation by Karl Müller is Claudii Ptolemaei Geographia, 2 vols. Paris, 1883, 1891.

Geogr. Tables

Fuad I. Haddad and E. S. Kennedy, Geographical Tables of Medieval Islam, to appear.

Ginzel

F. K. Ginzel, Handbuch der mathematischen und technischen Chronologie, vol. 1, Leipzig, 1906.

Hādī Hasan

Hādī Ḥasan, A History of Persian Navigation, London, 1928.

Handy Tables

N. Halma, Commentaire de Théon d'Alexandrie sur les tables manuelles astronomiques de Ptolemée, in three parts, Paris, 1822-25. The geographical tables are in Part 1, pp. 109-131.

Heath

T. L. Heath, A History of Greek Mathematics, 2 vols., Oxford, 1921.

Hinz

Walther Hinz, Islamische Masse und Gewichte, Leiden, 1955.

Honigmann

Ernest Honigmann, Die Sieben Klimata . . . , Heidelberg, 1929.

Hourani

G. F. Hourani, Arab Seafaring in the Indian Ocean in Ancient and Early Medieval Times, Princeton, 1951.

Hudūd

V. Minorsky, Ḥudūd al-ʿĀlam, "The Regions of the World", A Persian Geography 372 A. H. - 982 A.D., Oxford, 1937.

India

Al-Bīrūnī, Kitāb fī taḥqīq mā li'l-Hind . . . , ed. by C. E. Sachau, London, 1888; ed. Hyderabad-Dn., 1958; transl. C. E. Sachau as "Alberuni's India", 2 vols., London 1910.

Irani

R. A. K. Irani, Arabic Numeral Forms, Centaurus, vol. 4 (1955), pp. 1-12.

K & F

E. S. Kennedy and Nazim Faris, The Solar Eclipse Technique of Yaḥyā b. abī Manṣūr, Journal for the History of Astronomy, 1 (1970), pp. 28-38.

al-Khujandī

Abū Maḥmūd Ḥāmid ibn al-Khiḍr al-Khujandī, Risāla fī taṣḥīḥ al-mayl wa arḍ al-balad . . . , the Arabic text edited by Louis Cheikho in Al-Mashriq (Beirut), 11 (1908), pp. 60-69.

Khwārizmī, Algebra

Muḥammad b. Mūsā al-Khwārizmī, Kitāb al-mukhtaṣar fi ḥisāb al-jabr wa al-muqābala, ed. and transl. F. Rosen, London, 1831.

Khwārizmī Zīj

H. Suter, Die astronomischen Tafeln des Muḥammed ibn Mūsā al-Khwārizmī . . . , Kgl. Danske Vidensk. Skrifter, 7. R., Hist. og filos. Afd. 3,1, Copenhagen 1914. English transl. by O. Neugebauer, The Astronomical Tables of al-Khwārizmī, Hist. Filos. Skr. Dans. Vid. Selsk. 4, no. 2, Copenhagen 1962.

Kieffer

J. S. Kieffer, Galen's Institutio Logica, English Translation, Introduction, and Commentary, Baltimore, 1964.

King

David A. King, The Astronomical Works of Ibn Yūnus. Unpublished doctoral dissertation, Yale University, 1972.

Kunitzsch

P. Kunitzsch, Untersuchungen zur Sternnomenklatur der Araber, Wiesbaden, 1961.

Langles

L. Langles, Cosmographie composée . . . par . . . Mohhammed

ben Ahmed ben Ayās, Notices et extraits des Mss. de la Bibliothèque National, vol. 8(1810), Paris.

L.-P.

Stanley Lane-Poole, The Mohammedan Dynasties, Paris, 1925.

Le Str.

G. Le Strange, The Lands of the Eastern Caliphate, Cambridge, 1930.

Maqālīd

E. S. Kennedy, Al-Bīrūnī's Maqālīd Ilm al-Hay'a, Journal of Near Eastern Studies 30(1971), pp. 308-314.

Marvazi

Sharaf al-Zamān Ṭāhir Marvazī on China, the Turks and India, ed., transl., and comm. by V. Minorsky, London, 1942.

Mas'ūdī, see Murūj

Merid.

E. S. Kennedy, Al-Bīrūnī on Determining the Meridian, The Mathematics Teacher, LVI(1963), pp. 635-637.

Merveilles

Mémorial Jean Sauvaget, vol. 1, Les merveilles de l'Inde, pp. 189-312, Damascus, 1954.

Meteorol.

Aristotle, Meteorologica, with an English trans. by H. D. P. Lee, the Loeb Classical Library, Cambridge Mass., 1952.

Meyerhoff

Max Meyerhoff, Von Alexandrien nach Baghdad. Ein Beitrag zur Geschichte des philosophischen und medizinischen Unterrichts bei den Arabern, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1930, Phil.-Hist. Klasse, pp. 389-429.

Morison

S. E. Morison, The Two-Ocean War. A Short History of the United States Navy in the Second World War, Boston, 1963.

Murūj

Maçoudi, Les prairies d'or, texte et trad. par C. Barbier de Meynard et Pavet de Courteille, vol. I, Paris, 1861.

Minorsky, see Ḥudūd
Marvazī

Mžik

Hans von Mžik unter Mitarbeitung von Friedrich Hopfner, Das Klaudios Ptolemaios Einführung in die darstellende Erdkunde, Erster Teil: Theorie und Grundlagen der darstellenden Erdkunde. (Klotho, Bd. 5), Wien, 1938.

Nalivkin

D. V. Nalivkin, The Geology of the U.S.S.R., New York, 1960.

Nallino, see Scritti

Newberry

Percy E. Newberry, Notes on Seagoing Ships, Journal of Egyptian Archaeology, 27(1941), pp. 64-6.

Oppolzer

Th. Oppolzer, Canon der Finsternisse, Akad. d. Wiss., Wien Math.-Nat. Cl., Denkschriften 52, 1887. Stechert reprint, New York, 1921. Transl. by Owen Gingerich as Canon of Eclipses, Dover Publications, New York, 1962.

Optics

L'Optique de Claude Ptolémée dans la version latine d'ápres l'arabe de l'émir Eugène de Sicile. Ed. by Albert Lejeune, Louvain, 1956.

Overview

E. S. Kennedy, The History of Trigonometry, an Overview, pp. 333-359 in the 31st Yearbook of the National Council of Teachers of Mathematics, Washington D. C., 1969.

Pingree, see Fazārī
Ya'qūb

Posener

G. Posener, Le canal du Nil a la Mer Rouge avant les Ptolémées, Chronique d'Egypte, 13 (1938), pp. 259-273.

Ptolemy, see Almagest
Geogr.
Handy Tables
Tetrabiblos

Reinaud

M. Reinaud, Relation des voyages fait par les Arabes et les Persans dans l'Inde et à la Chine, vol. 1, Paris, 1845.

RT

A Russian translation and commentary to the Tahdīd by P. G. Bulgakov is: Abu Reihan, Izbrannye Proizvedeniya III

(Geodeziya), Akademiya Nauk Uzbekskoi SSR, Tashkent, 1966.

Sayili

A. Sayili, The Observatory in Islam, Ankara 1960.

Schirmer

Oskar Schirmer, Studien zur Astronomie der Araber, Sitzungsberichte der Physikalisch-medizinischen Sozietät zu Erlangen, 58 (1926), pp. 33-88.

Schoy, Alhazen

Carl Schoy, Abhandlung des al-Ḥasan ibn al-Ḥasan ibn al-Haitam über die Bestimmung der Richtung der Qibla, Zeitschrift der deutschen morgenländischen Gesellschaft, 75 (1921), pp. 242-253.

Schoy, Bestimmung

Karl Schoy, Bestimmung der geographischen Breite der Stadt Ghazna, mittels Beobachtungen im Meridian, durch den arabischen Astronomen und Geographen al-Bīrūnī, Annalen der Hydrographie und maritimen Meteorologie, 53 (1925), pp. 41-47.

Schoy, Kibla

Article by C. Schoy entitled "Ḳibla" in the EI, vol. 2, pp. 985-9.

Schoy, Nairīzī

C. Schoy, Abhandlung von al-Faḍl b. Ḥātim an-Nairīzī über die Richtung der Qibla, Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften, 1922, pp. 55-68.

Scritti

C. A. Nallino, Raccolta di Scritti editi e inediti, vol. 5, Rome, 1944.

Sedillot

L. A. Sédillot, Mémoire sur les instruments astronomiques des arabes, dans Mémoires presentes par divers savants, Academie des Inscriptions, Tome I, Paris, 1844.

Shadows

al-Bīrūnī, Ifrād al-maqāl fī amr al-ẓilāl, the second of four treatises published in a single volume under the title Rasāi'l al-Bīrūnī, Osmania Oriental Publications Bureau, Hyderabad-Dn., 1948. Engl. transl., unpubl., by E. S. Kennedy. References are to page and line of the published text.

Sharkas

E. S. Kennedy and Haydar Sharkas, Two Medieval Methods for finding the Obliquity of the Ecliptic, The Mathematics Teacher, 55 (1962), pp. 286-290.

Stein

Aurel Stein, "The Site of Alexander's Passage of the Hydaspes . . . , Geographical Journal, 80 (1932), pp. 31-46.

Strabo

The Geography of Strabo, text and transl. by H. L. Jones, vol. 1. The Loeb Classical Library, New York, 1917.

Survey

E. S. Kennedy, A Survey of Islamic Astronomical Tables, Trans. Am. Phil. Soc., N. S. 46, Pt. 2, Philadelphia, 1956.

Suter

H. Suter, Die Mathematiker und Astronomen der Araber und ihre Werke, Abhandlungen zur Geschichte der mathematischen Wissenschaften . . . , X Heft, Leipzig, 1900.

Tafhīm

al-Bīrūnī, Kitāb al-tafhīm li awā'il ṣinā'at al-tanjīm. The Arabic text with English translation was published by R. Ramsay Wright as "The Book of Instruction in the Art of Astrology", London, 1934. Bīrūnī's Persian version was published by Jalāl Humā'ī, Tehran, 1940.

Tahdīd

Al-Bīrūnī, Taḥdīd nihāyāt al-amākin li-taṣḥiḥ masāfāt al-masākin. The only extant copy is Istanbul MS Fatih 3386.

A criticial edition of the text by P. G. Bulgakov was published in a special issue of the Arab League journal Majallat ma'had al-makhṭūṭāt al-'arabiya, Cairo, 1962. An edition published at Ankara by Muḥammad al-Tanjī has been inaccessible to us.

The English translation, by Jamil Ali, is: The Determination of the Coordinates of Cities, al-Bīrūnī's, Taḥdīd al-Amākin, Centennial Publications, The American University of Beirut, Beirut, 1967.

For the Russian translation and commentary, see RT above.

Tarn

W. W. Tarn, The Greeks in Bactria and India, 2nd ed., Cambridge, 1951.

Tetrabiblos

Claudius Ptolemy, Tetrabiblos, ed. and transl. by F. E. Robbins, Harvard University Press (The Loeb Classical Library), 1940.

Thābit

O. Neugebauer, Thābit ben Qurra "On the Solar Year" and "On the Motion of the Eighth Sphere", Proc. Am. Phil. Soc., 106 (1962), pp. 264-299.

Thompson

M. W. Thompson, Novgorod the Great, Excavations at the Medieval City . . . , London, 1967.

Three Sages

Seyyed Hossein Nasr, Three Muslim Sages, Cambridge (Mass.), 1964.

Togan

A. Zeki Valīdī Togan, Bīrūnī's Picture of the World, Memoirs of the Archaeological Survey of India, No. 53, Delhi, 1940.

Traité

Naṣīr al-Dīn al-Ṭūsī, Traité du Quadrilatère, ed. and transl. by Alexandre Pacha Caratheodory, Constantinople, 1891.

Tuckerman

Bryant Tuckerman, Planetary, Lunar, and Solar Positions, A.D. 2 to A.D. 1649 . . . , Memoirs of the American Philosophical Society, Vol. 59, Philadelphia, 1964.

Wheeler

Geoffrey Wheeler, The Modern History of Soviet Central Asia, New York, 1964.

Wiedemann

Eilhard Wiedemann, Bestimmungen des Erdumfangs von al-Bērūnī, Archiv für die Geschichte der Naturwissenschaften und der Technik, 1 (1909), pp. 66–69.

Wolters

O. W. Wolters, Early Indonesian Commerce, Ithaca, 1967.

Ya'qūb

David Pingree, The Fragments of the Work of Ya'qūb ibn Ṭāriq, Journal of Near Eastern Studies, 26 (1960), pp. 97–125.

Here, and in the indices of parameters which follow, references consisting of a pair of numbers separated by a colon are to the page and line respectively of Bulgakov's edition of the Arabic text. The same pages and lines are indicated in Ali's English translation. Numbers preceded by an asterisk (*) are references to the numbered sections of this commentary. The Arabic definite article al, and ibn and abū have been ignored in alphabetizing. Some terms, such as sine, altitude, etc., occur so frequently that there is little point in noting each appearance. For each such word, a few occurrences have been entered below to enable readers to obtain the Arabic equivalent from the text, when desired.

INDEX OF SEXAGESIMAL PARAMETERS

Entries are arranged as in the other indices, and in the order of the leading non-zero digit, without regard to the location of the sexagesimal point, thus 1,6;40 precedes 1;48, and so on.

INDEX OF DECIMAL NUMBERS, DATES

Entries are arranged in the order of the leading digits, without regard to the location of the decimal point (e. g., 1016 precedes 75). As with the general index, references to the published text and English translation show page and line of the text, separated by a colon. References preceded by an asterisk are to sections of this commentary.

Misprints in the English Translation of the Taḥdīd

(Appreciation is expressed to Miss Janice Henderson, who communicated many of the following.)

Page	Line	
10	1, 4	for donkey read camel
17	4	for softs read soft
20	24	for three hundred read eight hundred
46	8	for on figure read one figure
65	4(from the bottom)	for Muḥarram read Ṣafar.
70	14	for by Sinān read by Ibrahim b. Sinān.
72	5(from the bottom)	for KT read HT.
74	4(from the bottom)	for GH, the maximum, read GH, the chord of twice the maximum.
75	3	for obtained 23;57, 45, 48, read obtained 47;55, 31, 35. Half of it is 23;57, 45, 48.
88	last line	for altitude read latitude.
89	5(from the bottom)	for its (LO's) read its (LC's)
96	21	for Tir read Shahrivar.
97	2	for 94;28° read 95;28°.
97	3	for 47;14°, is the latitude. read 47;44°, is the colatitude.
97	11	for tangent read cotangent.
105	9	for [5] 4;59, 59, 5 read [5]4;59, 59, 5.
111	23	for is the maximum, read is [twice] the sine of the maximum.
115	12	for 3, 813, 460, 925 read 3, 812, 460, 925.
117	9	for cosine of the sun's, read cosine of [half] the sun's.
117	15	for of the sun's read of [half] the sun's
124	14(from the bottom)	for WT read WZ.
125		On Figure 29 change the lower of the two X's to an S.
125	5	for TEC read YFC.
127		On Figure 30 mark the upper intersection between circles FSM and ABH with a W.
130	8(from the bottom)	for 1;48° read 1;49°.
132	11(from the bottom)	for fifty-six read twenty-four.
136	7(from the bottom)	for elpased read elapsed.

Page	Line	
145	4	for LF read SF, for ZM read KM.
145	5	for SFread SL, for KM read KZ.
165	13	for KT read HT.
169	12	for 7;5 read 7;0.
172	10(from the bottom)	for DHK read BHK.
173	15	for cosine of the equated latitude read cosine of the difference between the latitude of the locality (E) and the equated latitude.
181		The ninth entry under al-Farghānī is 0 9 31 56, it should be 0 9 31 46.
186	11	for BME read BMH.
194	19	for quared read squared.
194	last line	for 125, 664, 400 read 125, 664, 000.
228	7(from the bottom)	for 55;51, 58, 5, 42. read 55;51, 58, 5, 48.
233	2	for Rakhaj read Rukhkhaj.
242	2	for 26;22, 24°, read 27;22, 24°.
252	last line	for AHG read AEG.
253	2(from the bottom)	delete Its arc sine is 25;38, 17.
269	15	for Sunday read Saturday.
269	21	for noontide of the second Tuesday of read noontide of Tuesday, the second of

بإذن من جامعة بيروت الأمريكية

طبع في ٨٠ نسخة

نشر بمعهد تاريخ العلوم العربية والإسلامية
بفرانكفورت ـ جمهورية ألمانيا الاتحادية
طبع في مطبعة شتراوس، هيرشبرج، ألمانيا الاتحادية